NONHOMOGENEOUS HEAD CONDUCTION (CONTINUED)

This problem is a bit harder and won't come up on any exam, but I keep it here for the interested reader.

$$u_t = ku_{xx} - hu; \qquad u(0,t) = 0, \quad u(\pi,t) = u_0; \qquad u(x,0) = 0. \tag{1}$$

Solution: To get the equilibrium solution we solve

$$ku_{xx} - hu = 0 \Rightarrow u_* = C_1 \cosh\left(\sqrt{\frac{h}{k}}x\right) + C_2 \sinh\left(\sqrt{\frac{h}{k}}x\right)$$

The first boundary gives us $u_*(0)C_1 = 0$ and the second gives us

$$u_*(\pi) = C_2 \sinh\left(\sqrt{\frac{h}{k}}\pi\right) = u_0 \Rightarrow C_2 = \frac{u_0}{\sinh\left(\sqrt{\frac{h}{k}}\pi\right)} \Rightarrow u_* = u_0 \frac{\sinh\left(\sqrt{\frac{h}{k}}x\right)}{\sinh\left(\sqrt{\frac{h}{k}}\pi\right)}$$

Letting $v(x,t) = u(x,t) - u_*(x)$ gives us

$$v_t = kv_{xx} - hv; \qquad v(0,t) = v(\pi,t) = 0; \qquad v(x,0) = -u_0 \frac{\sinh\left(\sqrt{\frac{h}{k}}x\right)}{\sinh\left(\sqrt{\frac{h}{k}}\pi\right)} \tag{2}$$

HEAT IN NONUNIFORM MEDIA

Consider the 1-D problem

$$\rho(x)c(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K(x)\frac{\partial u}{\partial x}\right) + Q(x,t). \tag{3}$$

To simplify the problem slightly, let $Q(x,t) = \alpha u(x,t)$.

There is no guarantee that separation of variables will work, but lets try it and see what we get. Let $u(x,t) = X(x)T(t)$. Then

$$\rho(x)c(x)T'X = \frac{\partial}{\partial x}\left(K(x)X'\right)T + \alpha XT$$
$$\frac{T'}{T} = \frac{1}{c\rho X}\frac{d}{dx}\left(K\frac{dX}{dx}\right) + \frac{\alpha}{c\rho} = -\mu$$

where μ is the eigenvalue of the problem. Solving for T gives us the usual $T = e^{-\mu t}$. Our X equation becomes

$$\frac{d}{dx}\left(KX'\right) + (\alpha + \mu c\rho)X = 0. \tag{4}$$

This is called a Sturm–Liouville Eigenvalue Problem, which takes the general form

$$\frac{d}{dx}\left(p\frac{dy}{dx}\right) + qy + \lambda\sigma y = 0. \tag{5}$$

It is reasonable to assume that y can be expressed as a Fourier-like series. What will change are the coefficients and eigenfunctions. Notice that with the simple heat problem the general solution has linearly independent solutions in the form of sine and cosine. Here we let ϕ_n represent general linearly independent solutions to the general Sturm–Liouville problem.

Consider (5) with solutions ϕ_n. Notice that since they are linearly independent they must satisfy the Wronskian being zero

$$\begin{vmatrix} \phi_i(a;\lambda) & \phi_j(a;\lambda) \\ \phi_i(b;\lambda) & \phi_j(b;\lambda) \end{vmatrix} = 0. \tag{6}$$

Since ϕ_i and ϕ_j are linearly independent solutions,

$$\frac{d}{dx}(K\phi_i') + (q + \lambda_i\sigma)\phi_i = 0, \text{ and}$$
$$\frac{d}{dx}(K\phi_j') + (q + \lambda_j\sigma)\phi_j = 0.$$

Multiplying the first by ϕ_j and the second by ϕ_i and then subtracting gives us

$$\phi_j\frac{d}{dx}(K\phi_i') - \phi_i\frac{d}{dx}(K\phi_j') + (\lambda_i - \lambda_j)\sigma\phi_i\phi_j = 0.$$

We can integrate the first two expressions over the domain $[a, b]$ by parts to get

$$\int_a^b \left[\phi_j(K\phi_i')' - \phi_i(K\phi_j')'\right] dx = 0,$$

which means

$$(\lambda_i - \lambda_j)\int_a^b \sigma\phi_i\phi_j dx = 0. \tag{7}$$

So, if $\lambda_i \neq \lambda_j$,

$$\int_a^b \sigma\phi_i\phi_j dx = 0, \tag{8}$$

which makes that integral an orthogonality condition. We can use this orthogonality in the same way as we did that of Fourier series, so

$$f(x) = \sum_{n=0}^{\infty} c_n\phi_n \Rightarrow c_n = \int_a^b \sigma(x)\phi_n f(x)dx. \tag{9}$$

6.1 Basic simplifications and its shortcomings

Now lets shift gears and look at how to simplify equations, which can be very useful when working with complex equations. Lets see what happens if we remove small terms and check for consistency.

Ex: Consider $x + 10y = 21$ and $5x + y = 7$. Suppose that we decide x is small compared to $10y$ and 21. If we remove x from the first equation, we get $y \sim 2.1$, then $x \sim (7 - 2.1)/5 \sim 0.98$. Notice that the actual solution is $x = 1$ and $y = 2$, so this approximation is not far off.

Ex: Consider the ODE for a projectile,

$$x'' = -\frac{gR^2}{(x + R)^2};\ x(0) = 0,\ x'(0) = v, \tag{10}$$

where R is the radius of the Earth, x is the radial distance of the projectile from the ground, g is acceleration due to gravity at the surface, and v is the initial velocity. If $v \ll 1$, $x \ll R$; i.e., $(x + R)^2 \sim R^2$. Then our simplified equation becomes

$$x'' = -g; x(0) = 0, x'(0) = v \Rightarrow x = -\frac{1}{2}gt^2 + vt \Rightarrow x' = -gt + v. \tag{11}$$

Notice that if $x' = 0$, $t = v/g$, which gives us the maximum height of $x_{\max} = v^2/2g$. Then $\max(x/R) = v^2/2gR$. Observe that $x \ll R$ implies $\max(x/R) \ll 1$, so if $v^2 \ll gR$ this solution is consistent.

Ex: Consider $0.01x + y = 0.1$ and $x + 101y = 11$. If we assume $0.01x$ is much smaller than y, then $y \sim 0.1$ and $x \sim 11 - 101 \cdot 0.1 = 99$. However, the actual solutions are $y = 1$ and $x = -90$, so this approximation is off by a factor of 10 for y and 100 for x, and a minus sign. Obviously, this approximation is not consistent.

Nondimensionalization. Consider the projectile problem again,

$$\ddot{x} = -\frac{gR^2}{(x+R)^2};\ x(0) = 0,\ \dot{x}(0) = v. \tag{12}$$

Lets first try $\hat{x} = x/R$ and $\tau = t/T$. Our derivatives are

$$\frac{dx}{dt} = \frac{dx}{d\hat{x}}\frac{d\hat{x}}{d\tau}\frac{d\tau}{dt} = \frac{R}{T}\frac{d\hat{x}}{d\tau},$$
$$\frac{d^2x}{dt^2} = \frac{d\dot{x}}{d\hat{x}'}\frac{d\hat{x}'}{d\tau}\frac{d\tau}{dt} = \frac{R}{T^2}\frac{d^2\hat{x}}{d\tau^2}.$$

This gives us the ODE

$$\frac{R}{T^2}\frac{d^2\hat{x}}{d\tau^2} = -\frac{gR^2}{(R\hat{x}+R)^2} = -\frac{g}{(\hat{x}+1)^2};\ \hat{x}(0) = 0,\ \hat{x}'(0) = \frac{T}{R}\dot{x}(0) = \frac{T}{R}v. \tag{13}$$

Since R/v is in units of time, a natural guess for T would be $T = R/v$. Then we get $R/gT^2 = v^2/gR = \varepsilon \ll 1$ if $R \gg v^2/g$. However this would give us $\varepsilon\hat{x}''$, which means to leading order we would be getting rid of the most important term (the higher the order of the derivative the more important the term).

Recall from last time we had that another guess for T is $T = \sqrt{R/g}$. Then

$$\hat{x}'(0) = \frac{T}{R}\dot{x}(0) = \frac{v}{\sqrt{gR}} = \sqrt{\varepsilon}$$

if $R \gg v^2/g$. However, this won't work either since this means $\hat{x}'(0) = 0$ to leading order.

So, it's not the choice of T that's a problem, but rather the choice of $\hat{x}$. Recall from last time that the maximum height the projectile can reach, $x_{\max} = v^2/2g$ is much less than the radius of the Earth. So, lets scale by that.

Let $\hat{x} = (g/v^2)x$ and $\tau = t/T$. Then

$$\hat{x}' = \frac{g}{v^2T}\dot{x} \text{ and } \hat{x}'' = \frac{g}{v^2T^2}\ddot{x}.$$

Our ODE becomes

$$\frac{v^2T^2}{g}\hat{x}'' = \frac{-g}{\left(\frac{v^2}{gR}\hat{x}+1\right)^2} \Rightarrow \frac{v^2T^2}{g^2}\hat{x}'' = \frac{-1}{\left(\frac{v^2}{gR}\hat{x}+1\right)^2}.$$

We would like $\hat{x}''$ to remain so we let $T = g/v$ and $\varepsilon = v^2/gR$, then the full nondimensionalized ODE is

$$\ddot{x} = -\frac{1}{(1+\varepsilon x)^2};\ x(0) = 0,\ \dot{x}(0) = 1. \tag{14}$$