Math 5346 Rahman

Week 2

1.3 Aggregation of Slime Mold Amebae

For simplicity lets consider 1-D aggregation. We looked at the derivation in more detail in class, but here I will simply write down the main calculations.

First let $a(x,t)$ be the number of amebae at a position x and time t. Consider a small region $[x, x + \Delta x]$. Let $J(x,t)$ and $J(x + \Delta x, t)$ be the flux of amebae at the boundaries x and $x + \Delta x$. Further, let $Q(x,t)$ be the net creation of amebae inside the region.

While we are interested in the number of amebae at a specific point, we need to first think of a small region and then take a limit. To get the rate of change of the number of amebae in our region we take the integral. Then this will be equivalent to the flux in plus the integral of amebae production in the region,

$$\frac{\partial}{\partial t} \int_x^{x+\Delta x} a(\bar{x},t)d\bar{x} = J(x,t) - J(x + \Delta x, t) + \int_x^{x+\Delta x} Q(\bar{x},t)d\bar{x}. \tag{1}$$

Then by the Mean Value Theorem we get

$$\frac{\partial}{\partial t}[a(\xi,t)\Delta x] = J(x,t) - J(x + \Delta x, t) + Q(\xi,t)\Delta x \tag{2}$$

for $\xi \in [x, x + \Delta x]$. Dividing through by Δx gives us

$$\frac{\partial}{\partial t}a(\xi,t) = \frac{J(x,t) - J(x + \Delta x, t)}{\Delta x} + Q(\xi,t). \tag{3}$$

Then we take the limit $\Delta x \to 0$,

$$\frac{\partial a}{\partial t}(x,t) = -\frac{\partial J}{\partial x}(x,t) + Q(x,t). \tag{4}$$

Now lets think of possible simplifications and how to characterize the flux. If the birth and death rate is equivalent, then we have $Q(x,t) = 0$. For flux the amount of amebae entering a boundary is inversely proportional to the amount of amebae present in the region; i.e., if there is more amebae in a region, other amebae won't want to enter in order to avoid overcrowding. Then the flux is

$$J(x,t) = -\mu\frac{\partial a(x,t)}{\partial x}. \tag{5}$$

This gives us the simplest possible model for amebae aggregation

$$\frac{\partial a}{\partial t} = \mu\frac{\partial^2 a}{\partial x^2}. \tag{6}$$

Notice that this is just the diffusion equation, and we solve this via separation of variables. If, however, we have $Q \neq 0$, then we would need to homogenize the problem before using separation of variables. In order to homogenize we would need to find the steady-state of the equation and then do a change of variables. If you are interested and haven't seen this before, you can look it up in most PDE textbooks or online by searching "homogenizing the heat equation".

After formulating the simplest possible model, we can think of what complications can be added in order to make deeper scientific inquiries. Lets suppose there are attractants present in our domain. Let ρ represent the density of attractants. This will add to the flux. If the attractant concentration in a region is higher the amebae will go towards it; further, if there is a lot of amebae in a region, the flux will be increased towards the attractants. Then the complete flux can be written as

$$J(x,t) = -\mu\frac{\partial a(x,t)}{\partial x} + \chi a\frac{\partial \rho(x,t)}{\partial x}, \tag{7}$$

and hence the model becomes

$$\frac{\partial a}{\partial t} = \frac{\partial}{\partial x}\left(\mu\frac{\partial a}{\partial x} - \chi a\frac{\partial \rho}{\partial x}\right). \tag{8}$$

However, the attractants themselves are moving around, and will have its own diffusion model

$$\frac{\partial \rho}{\partial t} = fa - k\rho + D\frac{\partial^2 \rho}{\partial x^2}. \tag{9}$$

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We have hit a point where in general the usual techniques learned in undergraduate courses won't work. So, we need to use numerics or stability theory. Stability is much easier to study with ODEs than PDEs, so lets do that.

Basics of Dynamical Systems

Consider plants in a small garden. Let $x = 0$ represent an empty garden and $x = 1$ represent a "full" one. Suppose that the birth rate is equivalent to the population size, and the death rate is equivalent to the square of the population size. We can model this as

$$\dot{x} = f(x) = x - x^2. \tag{10}$$

First we wish to find the fixed points (also called equilibrium solutions) as these are the easiest solutions to solve for. In order to do so, we simply let $\dot{x} = 0$ and solve for x to get $x_* = 0, 1$. Now we can draw the vector field, as done in class. Notice that we allow $x > 1$, but this means the garden is over capacity and plants die off due to overcrowding.

While graphical means for stability analysis is useful, it is often quicker to use calculus. By taking the derivative we get

$$f'(x_*) = 1 - 2x_* \Rightarrow f'(0) > 0, f'(1) < 1$$

so $x_* = 0$ is unstable and $x_* = 1$ is stable. However, this fails for fixed points where $f'(x_*) = 0$. This special case is called a nonhyperbolic fixed point. For our problem, both fixed points are hyperbolic. For nonhyperbolic fixed points, we need to study $f(x + \delta)$ as we briefly did in class.

Next time we shall look at what happens when we have two competing species.