Basics of Dynamical Systems (continued)

Consider further an animal that eats the plants at a 1:1 rate is introduced into the system. Suppose that the birth rate of the animal is equivalent to the population size of the plants, and the death rate is equivalent to its own population size. Further, assume the animal's feces acts like a fertilizer, effectively doubling the plant's population growth rate. We can model this as follows,

$$\dot{x} = 2x - x^2 - y, \qquad \dot{y} = x - y. \tag{1}$$

To find the fixed points we let $\dot{x} = 0$ and $\dot{y} = 0$,

$$\dot{x} = 0 \Rightarrow y = 2x - x^2$$

and

$$\dot{y} = 0 \Rightarrow y = x \Rightarrow x^2 - x = x(x-1) = 0,$$

then the fixed points are $(x_*, y_*) = (0,0)$, $(1,1)$.

You may have noticed that letting $\dot{x} = 0$ and $\dot{y} = 0$, produced two lines: $y = 2x - x^2$ and $y = x$. These are called <u>nullclines</u>. These are lines where there is no motion in the x-direction for $\dot{x} = 0$ and no motion in the y-direction for $\dot{y} = 0$. Sometimes it is useful to plot these lines, and sometimes they are completely useless. Whether we need them will depend on the application.

Now we compute the Jacobian in order to find the eigenvalues and eigenvectors for the respective fixed points. The general Jacobian for this system is

$$J(x_*, y_*) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 2 - x_* & 1 \\ 1 & -1 \end{pmatrix} \tag{2}$$

Then we have

$$J(0,0) = \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}, \qquad J(1,1) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \tag{3}$$

Then we compute the eigenvalues

$$|J(0,0) - \lambda I| = \begin{vmatrix} 2 - 2\lambda & -1 \\ 1 & -1 - \lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0 \Rightarrow \lambda = \frac{1}{2} \pm \frac{\sqrt{5}}{2} \tag{4}$$

and

$$|J(1,1) - \lambda I| = \begin{vmatrix} -\lambda & -1 \\ 1 & -1 - \lambda \end{vmatrix} = \lambda^2 + \lambda + 1 = 0 \Rightarrow \lambda = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \tag{5}$$

So, $(0,0)$ is a saddle and $(1,1)$ is a stable spiral.

Then the respective eigenvectors are

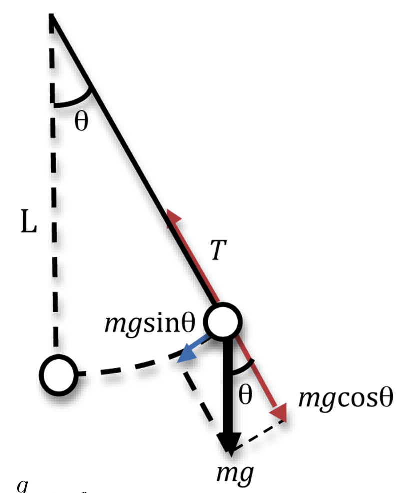
$$\begin{pmatrix} \frac{3}{2} \mp \frac{\sqrt{5}}{2} & -1 \\ 1 & -\frac{3}{2} \mp \frac{\sqrt{5}}{2} \end{pmatrix} v = 0 \Rightarrow v = \begin{pmatrix} 1 \\ \frac{3}{2} \mp \frac{\sqrt{5}}{2} \end{pmatrix} \tag{6}$$

and

$$\begin{pmatrix} \frac{1}{2} \mp i\frac{\sqrt{3}}{2} & -1 \\ 1 & -\frac{1}{2} \mp i\frac{\sqrt{3}}{2} \end{pmatrix} v = 0 \Rightarrow v = \begin{pmatrix} 1 \\ \frac{1}{2} \mp i\frac{\sqrt{3}}{2} \end{pmatrix} \tag{7}$$

Sometimes we just can't solve problems exactly. In fact most problems don't have exact solutions. These equations are either solved numerically, in which case the numerics might get it wrong, or we extract information form the equations without solving.

Consider the pendulum. We will consider both the frictional and frictionless case. **For the exam just to keep things simple we will just focus on the frictionless case**, but it is important to understand what happens in the frictional case as well.



In order to derive the ODE we need the force along its arc. Then we use Newton's law: $F = ma$. To find the acceleration a along the arc lets consider the arc length: $s = L\theta$, then the velocity is $v = L\frac{d\theta}{dt} = \dot{\theta}$, and the acceleration is $a = L\frac{d^2\theta}{dt^2} = \ddot{\theta}$, which gives us a force of $F = m\ddot{\theta}$. Now we need to figure out what F is. This consists of the force from gravitational acceleration and damping from friction, $F = -\nu L\dot{\theta} - mg\sin\theta$. This gives us the ODE

$$mL\frac{d^2\theta}{dt^2} = -\nu L\frac{d\theta}{dt} - mg\sin\theta \Rightarrow \frac{d^2\theta}{dt^2} + \frac{\nu}{m}\frac{d\theta}{dt} + \frac{g}{L}\sin\theta = 0$$

But this is really ugly, so lets simplify the equation a bit,

$$\ddot{\theta} = -\gamma\dot{\theta} - k\sin\theta \tag{8}$$

Higher order equations are tough to deal with, especially when they are nonlinear, so lets change this into a system of first order equations by letting $\omega = \dot{\theta}$

$$\begin{aligned}\dot{\theta} &= \omega \\ \dot{\omega} &= -\gamma\omega - k\sin\theta\end{aligned}$$

We can't change this into a matrix equation because it is nonlinear. However, we can use similar techniques after linearizing about special solutions called *fixed points* (f.p.'s) (θ_*, ω_*), which are points for which the object isn't moving; i.e. $\dot{\omega} = 0$ and $\dot{\theta} = 0$. This means we set the right hand side (RHS) to zero

$$\dot{\theta}_* = 0 \Rightarrow \omega_* = 0 \Rightarrow \dot{\omega_*} = -\gamma\overset{0}{\omega_*} - k\sin\theta_* = 0 \Rightarrow \theta_* = n\pi;\ n \in \mathbb{Z} \Rightarrow (\theta_*, \omega_*) = (n\pi, 0)$$

Now we can linearlize about these fixed points. In order to do this we will use the *Jacobian* also known as the derivative matrix in 2-D.

$$J(\theta_*, \omega_*) = \begin{pmatrix} \frac{\partial f}{\partial \theta} & \frac{\partial f}{\partial \omega} \\ \frac{\partial g}{\partial \theta} & \frac{\partial g}{\partial \omega} \end{pmatrix}_{(\theta_*, \omega_*)} \tag{9}$$

For our case this will be

$$J(n\pi, 0) = \begin{pmatrix} 0 & 1 \\ -k\cos\theta & -\gamma \end{pmatrix}_{(n\pi,0)} = \begin{pmatrix} 0 & 1 \\ \mp k & -\gamma \end{pmatrix} \text{ for } n \text{ even and odd, respectively.} \tag{10}$$

Intuitively we know that we are going to get different solutions for the frictional and frictionless case. Lets first do the frictionless case.

Frictionless Case ($\gamma = 0$)**.** Here the fixed points are the same, but now our Jacobian is going to be slightly different

$$J(n\pi, 0) = \begin{pmatrix} 0 & 1 \\ -k\cos\theta & 0 \end{pmatrix}_{(n\pi,0)} = \begin{pmatrix} 0 & 1 \\ \mp k & 0 \end{pmatrix} \text{ for } n \text{ even and odd, respectively.} \tag{11}$$

Then we look for the eigenvalues for our two fixed point cases.

$\underline{n \text{ even.}}$ When n is even we have

$$\begin{vmatrix} -\lambda & 1 \\ -k & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 = -k \Rightarrow \lambda = \pm\sqrt{-k} = \pm i\sqrt{k} \tag{12}$$

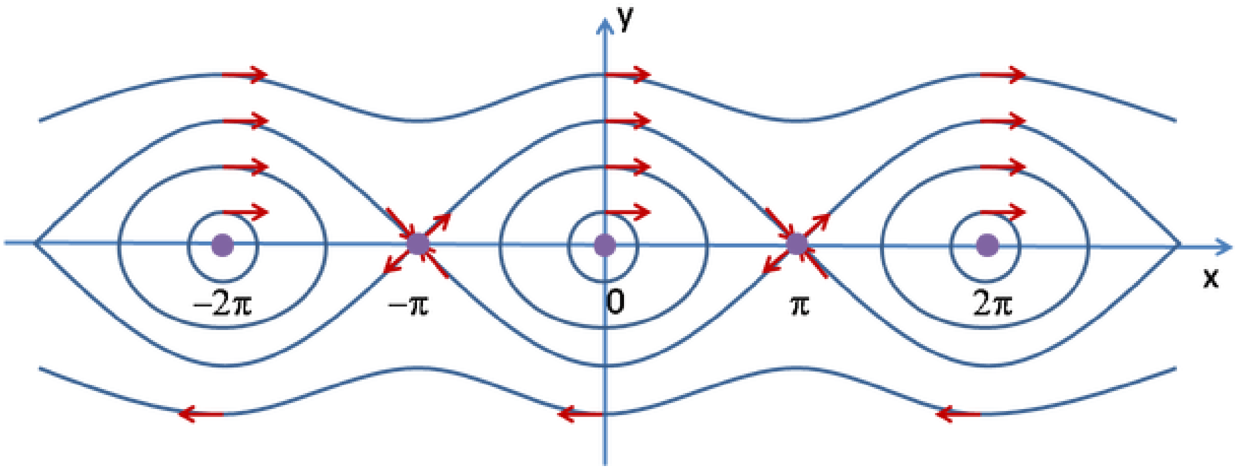
This is what's called a *center* fixed point, and this admits oscillatory solutions since we have complex conjugate eigenvalues without a real part; i.e. pure sines and cosines.

<u>n odd.</u> When n is odd we have

$$\begin{vmatrix} -\lambda & 1 \\ k & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 = k \Rightarrow \lambda = \pm\sqrt{k} \tag{13}$$

This is called a *saddle* fixed point, and this admits exponentially decaying solutions in one direction and exponentially expanding solutions in the other direction.

Now we can sketch what's called a *phase portrait* also called a *phase plane diagram.* This is a vector field in (θ, ω) space. This is a way we can visualize how the position and velocity of solutions are related. The phase portrait for the frictionless pendulum is going to look as such When two saddle fixed points are



connected it is called a *heteroclinic orbit.* And if a saddle is connected to itself it's called a *homoclinic orbit.* Attached to the email are videos of the different solutions about the center fixed point getting closer to the heteroclinic orbit. The video doesn't, however, show the heteroclinic solution because it takes infinitely long to go from one saddle point to the other. It also doesn't show the solutions away from the heteroclinic orbit, which correspond to a pendulum with so much initial energy that it just moves around in circles forever.

Next time I will show the frictional case.