PHASE PLANES (CONTINUED)

Frictional Case ($\gamma > 0$). Now lets look at the frictional case. For this case I won't sketch the full phase portrait. I will just show the local vector field around the fixed point just so I can define the various types of stability. From (**??**) we have two cases of n. Lets first look at the odd n's

<u>n odd.</u> When n is odd we have

$$\begin{vmatrix} -\lambda & 1 \\ k & -\gamma-\lambda \end{vmatrix} = \lambda^2 + \gamma\lambda - k = 0 \Rightarrow \lambda = \frac{1}{2}\left(-\gamma \pm \sqrt{\gamma^2+4k}\right). \tag{1}$$

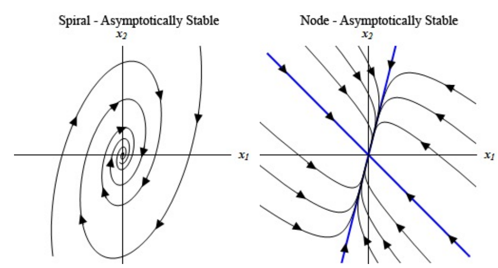
Notice that the discriminant here is always positive, also $\sqrt{\gamma^2+4k} > \gamma$, which means for $+\sqrt{\gamma^2+4k}$ it is exponentially expanding and for $-\sqrt{\gamma^2+4k}$ it is exponentially decaying. So the fixed point when n is odd is a *saddle* fixed point. We already know the sketch of a saddle so I won't sketch it here.

<u>n even.</u> When n is even we have

$$\begin{vmatrix} -\lambda & 1 \\ -k & -\gamma-\lambda \end{vmatrix} = \lambda^2 + \gamma\lambda + k = 0 \Rightarrow \lambda = \frac{1}{2}\left(-\gamma \pm \sqrt{\gamma^2-4k}\right). \tag{2}$$

Notice that this has two cases. Either the discriminant is negative or positive.

<u>$\gamma^2 - 4k < 0$.</u> For this case we get complex conjugate eigenvalues, but notice that the real part is negative. This means we will get decaying oscillatory solutions, and the fixed point is called a *stable spiral*.



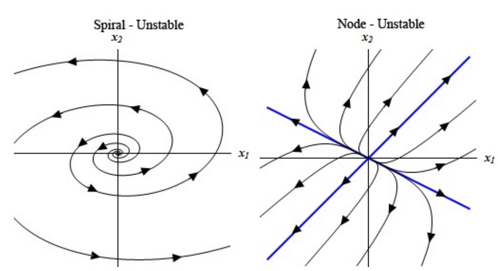
<u>$\gamma^2 - 4k > 0$.</u> For this case we get negative real solutions since $\sqrt{\gamma^2-4k} < \gamma$. This means we get decaying solutions, and the fixed point is called a *stable node*.

Forced Case ($\gamma < 0$). There is one more case. What if γ is negative. Then this isn't damped anymore, it's actually forced. We still have the even and odd cases, but now the real part will always be positive.

<u>n odd.</u> When n is odd we have a saddle fixed point again since we have both positive and negative real eigenvalues because $\sqrt{\gamma^2+4k} > \gamma$, so for $+\sqrt{\gamma^2+4k}$ it is exponentially expanding and for $-\sqrt{\gamma^2+4k}$ it is exponentially decaying.

<u>n even.</u> When n is even we have the same positive or negative discriminant, except now we have positive real parts.

<u>$\gamma^2 - 4k < 0$.</u> For this case we get complex conjugate eigenvalues, but notice that the real part is positive. This means we will get expanding oscillatory solutions, and the fixed point is called an *unstable spiral*.



<u>$\gamma^2 - 4k > 0$.</u> For this case we get positive real solutions since $\sqrt{\gamma^2-4k} < \gamma$. This means we get expanding solutions, and the fixed point is called an *unstable node*.

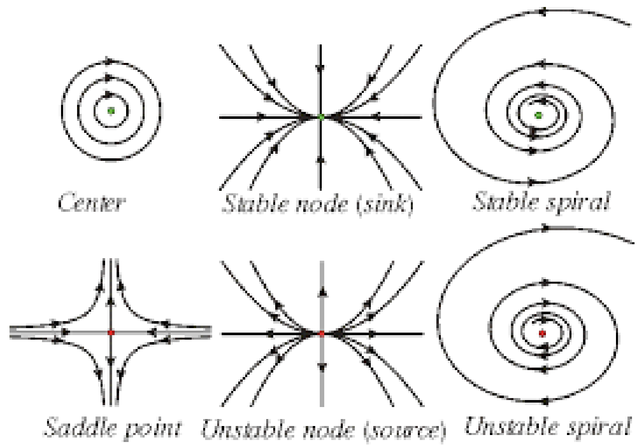
Summary. We can summarize all of this in this table. There are other cases called *borderline* cases, but lets not worry about that for this class.

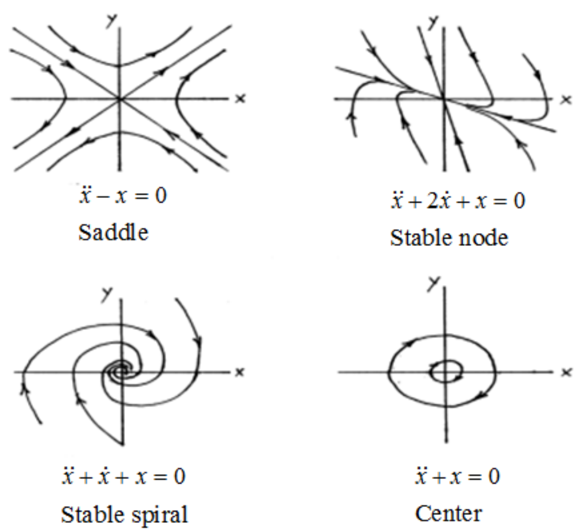
And here is another picture of the the types of fixed points when they don't line up perfectly with the axes:

Additional Examples.

Ex: Lets do a couple of really simple example that you will definitely not see on the exam just to solidify some concepts. These concepts in 1-D will expand to 2-D, but in 1-D we only have three cases, whereas in 2-D we have six main cases and three borderline cases. In this class we won't go over borderline cases.

Case	Eigenvalue	Description
Stable Node	$\lambda_{1,2} < 0$	Exponential Decay
Unstable Node	$\lambda_{1,2} > 0$	Exponential Expansion
Stable Spiral	$\lambda_{1,2} = \xi \pm i\theta$ where $\xi < 0$	Oscillatory Decay
Unstable Spiral	$\lambda_{1,2} = \xi \pm i\theta$ where $\xi > 0$	Oscillatory Expansion
Center	$\lambda_{1,2} = \pm i\theta$	Pure Oscillations
Saddle	$\lambda_1 < 0$ and $\lambda_2 > 0$	Exponential Decay in one direction and Exponential Expansion in the other





(a) Consider $\frac{dx}{dt} = \dot{x} = f(x) = -x$. Clearly the fixed point is $x = 0$, but is it stable or unstable? Well, if $x > 0$ the ODE will pull it back to zero, and if $x < 0$ it will also go back to zero. So this fixed point is *stable*. Another way we can show this is by taking the derivative, $f'(x_* = 0) = -1 < 0$.

(b) Now consider $\frac{dx}{dt} = \dot{x} = f(x) = x$. The fixed point is $x = 0$ again, but now it's *stable* since a point starting off the fixed point will want to go away from it. Also, $f'(x_* = 0) = 1 > 0$.

(c) We can also have something that is bi-stable. Consider $\frac{dx}{dt} = \dot{x} = f(x) = x^2$. Then for $x > 0$ the trajectory diverges from zero, but for $x < 0$ the trajectory goes towards zero. We'll notice that $f'(x_* = 0) = 0$, so linear stability analysis fails here; i.e. the derivative test is not enough to tell us anything about stability. Notice that $\frac{dx}{dt} = \dot{x} = f(x) = x^3$, also has a fixed point at zero with a zero derivative, but this is stable.

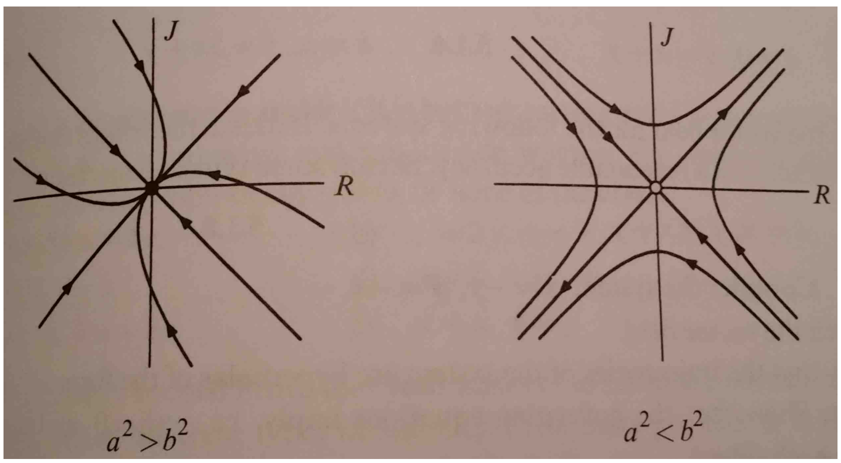
Ex: Now lets look at a 2-D example with linear equations. Fun fact: this is a model for love affairs written by Strogatz 1988.

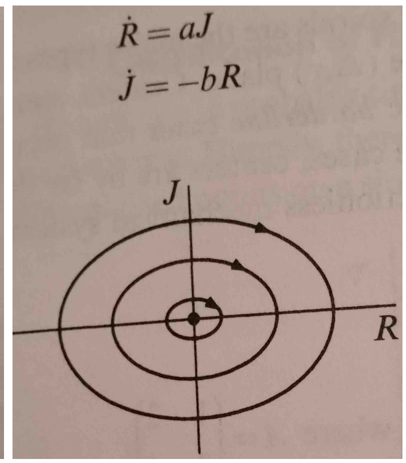
$$\begin{aligned}\dot{x} &= ax + by \\ \dot{y} &= bx + ay\end{aligned}; \qquad a < 0,\, b > 0 \tag{3}$$

Lets first find the fixed points: $ax_* + by_* = 0$; $cx_* + dy_* = 0 \Rightarrow x_* = y_* = 0$. You can convince yourself of this by either Gaussian elimination or simultaneous equations. Then we find the eigenvalues

$$J(x_*, y_*) = \begin{vmatrix} a & b \\ b & a \end{vmatrix} \Rightarrow \begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix} = \lambda^2 - 2a\lambda + a^2 - b^2 = 0 \Rightarrow \lambda = \frac{1}{2}\left(2a \pm \sqrt{4a^2 - 4a^2 + 4b^2}\right) = a \pm b.$$

Since $a < 0$ and $b > 0$ we have two cases: $|a| > |b|$: $\lambda_{1,2} < 0$ and $|a| < |b|$: $\lambda_1 = a - b < 0$, $\lambda_2 = b - a > 0$. These two cases are illustrated in the figures below (left). Now, we could use eigenvectors to get the precises direction of the trajectories, but these problems are simple enough that we can forgo using eigenvectors and just think of the vector field near the fixed point.





Ex: Lets look at a simpler version of the above equation

$$\begin{aligned}\dot{x} &= ay \\ \dot{y} &= -bx\end{aligned}; \qquad a, b > 0 \tag{4}$$

Clearly the fixed point is $(x_*, y_*) = (0, 0)$. For the eigenvalues we have

$$J(x_*, y_*) = \begin{vmatrix} 0 & a \\ -b & 0 \end{vmatrix} \Rightarrow \begin{vmatrix} -\lambda & a \\ -b & -\lambda \end{vmatrix} = \lambda^2 + ab = 0 \Rightarrow \lambda = \pm i\sqrt{ab}.$$

So this fixed point is a center with the phase portrait on the right.

Ex: Here is a concrete example somewhat similar to the one on the exam, except this problem is actually more difficult than the one that will be on the exam. The problem is exactly the way it will look on the exam except with a different ODE, so hope it helps.

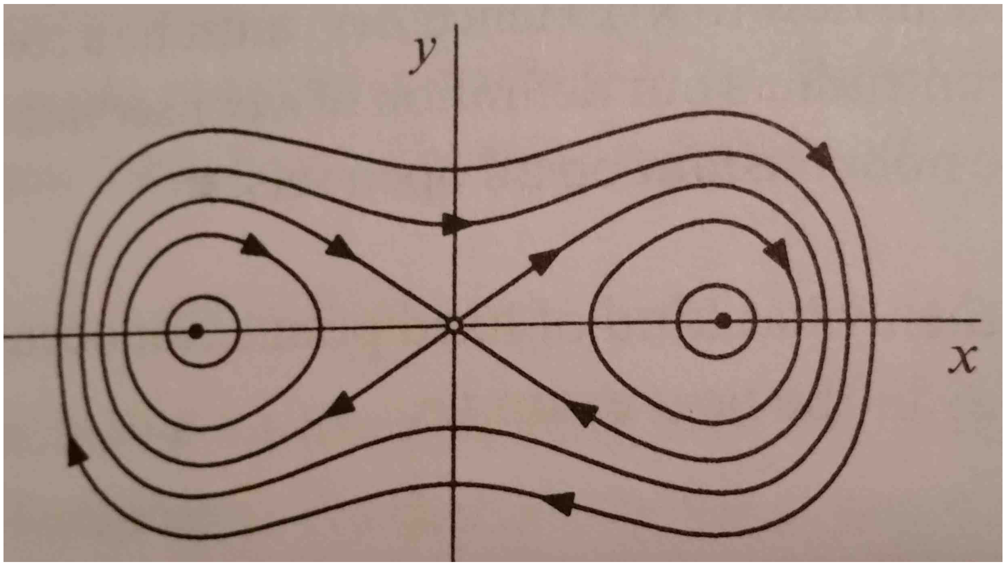
Consider the ODE

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= x - x^3\end{aligned} \tag{5}$$

(a) Find all fixed points.
Solution: $\dot{x}_* = 0 \Rightarrow y_* = 0$ and $\dot{y}_* = 0 \Rightarrow x_* - x_*^3 = x_*(1 - x_*^2) = x_*(1 - x_*)(1 + x_*) = 0 \Rightarrow x_* = 0, \pm 1$, so the three fixed points are $(x_*, y_*) = (0, 0)$, $(-1, 0)$, $(1, 0)$.

(b) Linearize about the fixed points.
Solution: First we compute the Jacobian,

$$J(x_*, y_*) = \begin{pmatrix} 0 & 1 \\ 1 - 3x_*^2 & 0 \end{pmatrix} \Rightarrow J(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad J(\pm 1, 0) = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$$



(c) Find the eigenvalues of the linearized system.
Solution: For $(x_*, y_*) = (0, 0)$ we have

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

And for $(x_*, y_*) = (\pm 1, 0)$ we get

$$\begin{vmatrix} -\lambda & 1 \\ -2 & -\lambda \end{vmatrix} = \lambda^2 + 2 = 0 \Rightarrow \lambda = \pm i\sqrt{2}$$

(d) State the stability of each fixed point.
Solution: $(x_*, y_*) = (0, 0)$ is a saddle fixed point since $\lambda_1 = -1 < 0$ and $\lambda_2 = 1 > 0$. $(x_*, y_*) = (\pm 1, 0)$ are centers since λ is a complex conjugate with zero real part.

(e) Sketch the phase portrait.
Solution: This has a homoclinic orbit unlike in the pendulum that had a heteroclinic orbit, because there is only one saddle point and the trajectory comes out of the saddle point back into the saddle point. The figure is below:

Sec. 2.2 Perturbation Theory

We can solve

$$\frac{dy}{dx} = 1 + (1+\varepsilon)y^2;\ y(0) = 0 \tag{6}$$

exactly via separation,

$$\frac{dy}{1 + (1+\varepsilon)y^2} = dx \Rightarrow \frac{1}{\sqrt{1+\varepsilon}} \tan^{-1}(\sqrt{1+\varepsilon}y) = x + C.$$

Invoking the initial condition gives us $C = 0$, then

$$y = \frac{\tan(x\sqrt{1+\varepsilon})}{\sqrt{1+\varepsilon}}. \tag{7}$$

Taking the Taylor series of this gives us

$$y = \tan x + \varepsilon \left(\frac{1}{2} x \sec^2 x - \frac{1}{2} \tan x \right) + \cdots \tag{8}$$

What if we couldn't find an exact solution though? Notice that the Taylor series of the exact solution above is of the form $y = \varepsilon^0 y_0 + \varepsilon^1 y_1 + \cdots$. Perhaps we can assume a solution of this form and test it for the problem that we already solved to verify that this method works.

Let

$$y = \varepsilon^0 y_0 + \varepsilon^1 y_1 + \varepsilon^2 y_2 + \cdots \tag{9}$$

Substituting this into (**??**) give us

$$\varepsilon^0 y_0' + \varepsilon^1 y_1' + \varepsilon^2 y_2' + \cdots = 1 + (1+\varepsilon)(\varepsilon^0 y_0 + \varepsilon^1 y_1 + \varepsilon^2 y_2 + \cdots)^2. \tag{10}$$

Now we need to separate this equation into its respective orders of ε; i.e., lets group terms that are of the same order of ε.

$o(\varepsilon^0)$: $y_0' = 1 + y_0^2$; $y_0(0) = 0 \Rightarrow y_0 = \tan x$.

$o(\varepsilon^1)$: $y_1' = y_0^2 + 2y_0 y_1 = \tan^2 x + (2\tan x)y_1$; $y_1(0) = 0$.

Another way we can write this is $y_1' - (2\tan x)y_1 = \tan^2 x$. For an ODE of this form we use integrating factors. Here the integrating factor is

$$\mu = \exp\left(\int^x -2\tan\xi d\xi\right) = \exp(2\ln\cos x) = \cos^2 x.$$

This gives us

$$d((\cos^2 x)y_1) = \cos^2 x \tan^2 x dx \Rightarrow (\cos^2 x)y_1 = \int \sin^2 x dx = \frac{1}{2}(x - \sin x\cos x) + C.$$

Invoking the initial condition once again gives us $C = 0$. Then

$$y_1 = \frac{1}{2}\left(x\sec^2 x - \tan x\right) \tag{11}$$

which is precisely the second term in the Taylor series of the exact solution (**??**).

Next time we apply this to the Pendulum.