MAPS

Maps are more natural models for certain physical systems, namely systems with natural recurrence. They can be easier to analyze sometimes. They are much easier to simulate (numerical analysis). ODEs require complicated - slow numerical schemes, and sometimes that doesn't even work. Maps only require loops carrying out a couple of arithmetic operations.

Definition 1. Maps are time discrete dynamical systems represented by the recurrence relation

$$x_{m+1} = f(x_m); x_m \in \mathbb{R}^n. \tag{1}$$

One great thing about maps is that we can create very simple examples of chaos since we aren't restricted by the dimensions of the dynamical system. Lets recall what some properties of maps are. We define a fixed point as $x_* = f(x_*)$. Due to the recurrent nature of maps, with this definition of a fixed point, the system cannot move away from that point once on it.

Consider $x_{n+1} = x_n^2$. To find the fixed points we do, $x_*^2 - x_* = 0$, so our fixed points are $x_* = 0, 1$.

For linear stability all eigenvectors corresponding to $|\lambda| < 1$ are the stable directions, all eigenvectors corresponding to $|\lambda| > 1$ are the unstable directions, and all eigenvectors corresponding to $|\lambda| = 1$ form the center subspace. It's easy to prove this in 1-D.

Proof. Consider $x_n = x_* + \xi_n$, then $x_* + \xi_{n+1} = x_{n+1} = f(x_* + \xi_n)$. We can find the Taylor series of f, $f(x + \xi_n) = f(x_*) + f'(x_*)\xi_n + o(\xi_n^2)$. Since $f(x_*) = x_*$, $\xi_{n+1} = f'(x_*)\xi_n + o(\xi_n^2)$. If ξ_n are small, $\xi_{n+1} \approx f'(x_*)\xi_n$. Let $\lambda = f(x_*)$. Now since $\xi_1 = \lambda\xi_0$ and $\xi_2 = \lambda\xi_1 = \lambda^2\xi_0$, by induction $\xi_n = \lambda^n\xi_0$, where ξ_0 is the initial point; i.e., just a constant. Therefore, if $|\lambda| < 1$, $\xi_n \to 0$ exponentially fast as $n \to \infty$; if $|\lambda| > 1$, $\xi_n \to 0$ exponentially fast as $n \to -\infty$; if $|\lambda| = 1$, ξ_n grows or decays subexponentially; i.e., $o(\xi_n^2)$ matter. □

Now consider the logistic map,

$$x_{n+1} = rx_n(1 - x_n); \qquad x_n \in [0, 1] \tag{2}$$

This comes from similar ideas to the logistic ODE, but $x_n = 1$ represents absolute capacity; i.e., if we have plants, there are so many that the soil becomes so devoid of nutrition that it can never sustain life again. The fixed points for this map are $x_* = (r - 1)/r, 0$. Notice that for $r \le 1$, there is only one fixed point since our $x_n \in [0, 1]$.

For linear stability we first take the derivative, $f'(x_*) = r - 2rx_*$. For $x_* = 0$, $f'(x_*) = r$, and the fixed point is stable for $r < 1$ and unstable for $r > 1$. For $f'(x_*) = 2 - r$, and $x_* = (r - 1)/r$, the fixed point is stable for $r < 3$ and unstable for $r > 3$. For the stable case there's no ambiguity, but for the unstable case we don't know in which way it is unstable, and for the borderline cases we don't even know the stability. In order to rectify this, we may use cobweb plots to help us illustrate the global behavior of a system at a glance. We did examples of cobwebs in class.

As with continuous systems, discrete systems experience bifurcations and contain periodic orbits. The analog of a periodic orbit for maps is a cycle.

Definition 2. We say a point $\hat{x}$ is contained in a k-cycle if $\hat{x} = f^k(\hat{x})$ and $\hat{x} \neq f^{k-1}(\hat{x})$, where f^k is the k^{th} iteration of $x_{n+1} = f(x_n)$.

Consider the map $x_{n+1} = 2 - x_n$, then $f^2(\hat{x}) = f(f(\hat{x})) = 2 - (2 - \hat{x})$. So every point for this map is in some 2-cycle. This is a trivial example, so lets look at the logistic map

$$f^2(x) - x = r^2x(1 - x)[1 - rx(1 - x)] - x = 0. \tag{3}$$

Factoring out x and $x - (r - 1)/r$ to remove the fixed points from the expression gives us

$$\hat{x} = \frac{1}{2r}\left[r + 1 \pm \sqrt{(r - 3)(r + 1)}\right]. \tag{4}$$

So these are the two points in the two cycle. Notice that for certain values of r the discriminant is negative, and hence $\hat{x}$ is not real. Therefore, for those values of r there is no two cycle.

For the logistic map, from $r = 3$, the fixed point starts to bifurcate into a 2-cycle, then a 4-cycle, then eventually to chaos. This is called a period doubling bifurcation.

A useful tool to quantify "sensitive dependence" is Liapunov exponents. If δ_0 is the initial separation, $|\delta_n| = |\delta_0| e^{n\lambda}$, then

$$\lambda = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|. \tag{5}$$