Fractals

Before we can talk about fractals we need to define some set theoretic ideas.

Cardinality of sets. We looked at some examples of finite sets in class, but what happens when two sets are infinite. Is one infinity bigger than the other? Notice that we can count Natural numbers, but we can't count say the Real numbers, so there must be a difference between these two sets.

Definition 1. A set $\mathbb{S}$ is said to be <u>countably infinite</u> if there is a one-to-one function from $\mathbb{N}$ to $\mathbb{S}$.

So the question we must ask when we are thinking about infinities is, can we write the elements of $\mathbb{S}$ as a sequence? Notice for $0, 1, 2, 3, \ldots$ our function is $f(n) = n - 1$ for $n \in \mathbb{N}$.

Definition 2. A set $\mathbb{S}$ is said to be <u>countable</u> if it is either finite or countably infinite, otherwise it is said to be <u>uncountable</u>.

Notice that the Reals are uncountable. To prove it takes a bit of effort, but we can intuitively see it when we try to "count" the reals.

Cantor set. A Cantor set is any set homeomorphic to $C = \prod_{n=1}^{\infty} F_n$, where each F_n is the two-point space $\{0, 1\}$. However, this is a complicated definition, and requires a knowledge of topology. Instead of defining it topologically, it is useful to construct the set and present pictoral examples.

The Cantor set is constructed from the set of Real numbers in the unit interval, $[0, 1]$. Let us call the initial set S_0. For the first iteration a fraction α is taken away from S_0 such that S_1 contains two disconnected sets of Real numbers $\left[0, \frac{1}{2}(1-\alpha)\right]$ and $\left[1 - \frac{1}{2}(1-\alpha), 1\right]$. Let $a = \frac{1}{2}(1-\alpha)$ and $b = 1 - \frac{1}{2}(1-\alpha)$, and call $[0, a]$, S_{1a} and $[b, 1]$, S_{1b}. For the second iteration, take away the fraction α from S_{1a} and S_{1b} such that S_2 contains four disconnected sets of Real numbers

$$\left[0, \frac{1}{2}(a - \alpha a)\right], \left[a - \frac{1}{2}(a - \alpha a), a\right], \left[b, b + \frac{1}{2}(a - \alpha a)\right], \text{ and } \left[1 - \frac{1}{2}(a - \alpha a), 1\right]. \tag{1}$$

This is continued *ad infinitum*, until S_∞ is reached, which is called the middle-α Cantor set partially shown in Fig. 1.

Figure 1. Example of set S_7.

For the sake of rigor, a closed form formula for each iteration is needed. It is easy to see that $S_1 = \frac{1}{2}(1-\alpha)S_0 \cup \left[\frac{1}{2}(1+\alpha) + \frac{1}{2}(1-\alpha)S_0\right]$. The formula for S_2, S_3, *etc* may seem somewhat difficult to derive, but with a little inspection and some computations a repeating pattern is seen. Notice $S_2 = \frac{1}{2}(1-\alpha)S_1 \cup \left[\frac{1}{2}(1+\alpha) + \frac{1}{2}(1-\alpha)S_1\right]$ and $S_3 = \frac{1}{2}(1-\alpha)S_2 \cup \left[\frac{1}{2}(1+\alpha) + \frac{1}{2}(1-\alpha)S_2\right]$. We may assume the formula for the n^{th} case follows this pattern.

$$S_n = \frac{1}{2}(1-\alpha)S_{n-1} \cup \left[\frac{1}{2}(1+\alpha) + \frac{1}{2}(1-\alpha)S_{n-1}\right]; \qquad 0 < \alpha < 1, \quad S_0 = [0, 1], \quad n \neq \infty. \tag{2}$$

The proof of n arbitrarily large is shown, however for $n = \infty$ a more rigorous proof is required.

Proof. By definition,

$$S_0 = [0, 1]. \tag{3}$$

It is easy to verify that

$$S_1 = \frac{1}{2}(1-\alpha)S_0 \cup \left[\frac{1}{2}(1+\alpha) + \frac{1}{2}(1-\alpha)S_0\right] = \left[0, \frac{1}{2}(1-\alpha)\right] \cup \left[1 - \frac{1}{2}(1-\alpha), 1\right]. \tag{4}$$

Suppose

$$S_n = \frac{1}{2}(1-\alpha)S_{n-1} \cup \left[\frac{1}{2}(1+\alpha) + \frac{1}{2}(1-\alpha)S_{n-1}\right]. \tag{5}$$

It can be shown by induction that

$$S_{n+1} = \frac{1}{2}(1-\alpha)S_n \cup \left[\frac{1}{2}(1+\alpha) + \frac{1}{2}(1-\alpha)S_n\right]. \tag{6}$$

□

When I first came up with this formula as an undergrad, I messed up the induction proof, and didn't realize it until now. It still works, but the proof will involve looking at the interval definition of S_{n+1}. Can you finish the induction?

Here are some important properties of Cantor sets

- Cantor sets are self-similar fractals: They look the same no matter what level you observe.
- They are completely disconnected: There are no intervals within a Cantor set (i.e., it has a topological dimension of zero, however it has a nonzero fractal dimension.)
- Cantor sets have a measure of zero: The length of S_n is α^∞. When $n \to \infty$ the length of S_n is zero because $0 < \alpha < 1$.
- Another way to find the measure is to subtract the measure of the complement of the Cantor set from the total length of $[0, 1]$. The length of the complement of the Cantor set is

$$\sum_{n=0}^{\infty} \left(\frac{1}{\alpha} - 1\right)^n \alpha^{n+1} = \alpha \sum_{n=0}^{\infty} \left(\frac{1}{\alpha} - 1\right)^n \alpha^n = \alpha \sum_{n=0}^{\infty} (1-\alpha)^n = \frac{\alpha}{\alpha} = 1. \tag{7}$$

Dimension of self-similar fractals. Notice that the Cantor set seems one-dimensional, but has more structure than the usual one-dimensional object. But of course, it's not two-dimensional. How do we rectify this? For a one-dimensional object, for example, we only have one copy and it's never scaled down. For self-similar fractals, at each iteration we make more copies and they are all scaled down. Take the Cantor set for example, we make $n = 2$ copies and scale them down by $r = 3$, so we can relate how the copies scale down as $n = r^d$, then $d = \ln(n)/\ln(r)$, then the dimension of the Cantor set is $d = \ln(2)/\ln(3)$, so it's between one and two dimensions.

In class we also discussed the von Koch curve shown in Fig. 2. Notice that this makes $n = 4$ copies and scales by a factor of $r = 3$, so it will be between one and two dimensions: $d = \ln(4)/\ln(3)$.

Box dimension. The idea of the box dimension is to cover the entire set with the minimum number of boxes of size ε. Let N bet the number of boxes, then $d = \lim_{\varepsilon \to 0} \ln(N)/\ln(1/\varepsilon)$. This works well for, say non-self-similar versions of the Sierpinski carpet (Fig. 3), but rarely works in practice for general fractals.

Fractals in chaos. Phase space chaotic systems very typically have fractal structure. For example, strange attractors, horseshoes, period doubling and others all have fractal structure. One way we can think of the dimension of an object is the number of times a trajectory intersects balls around various points. This is called the correlation dimension. However, this can only give us an approximation. There are also sets where the dimension varies from one region to another, which are called multifractals.

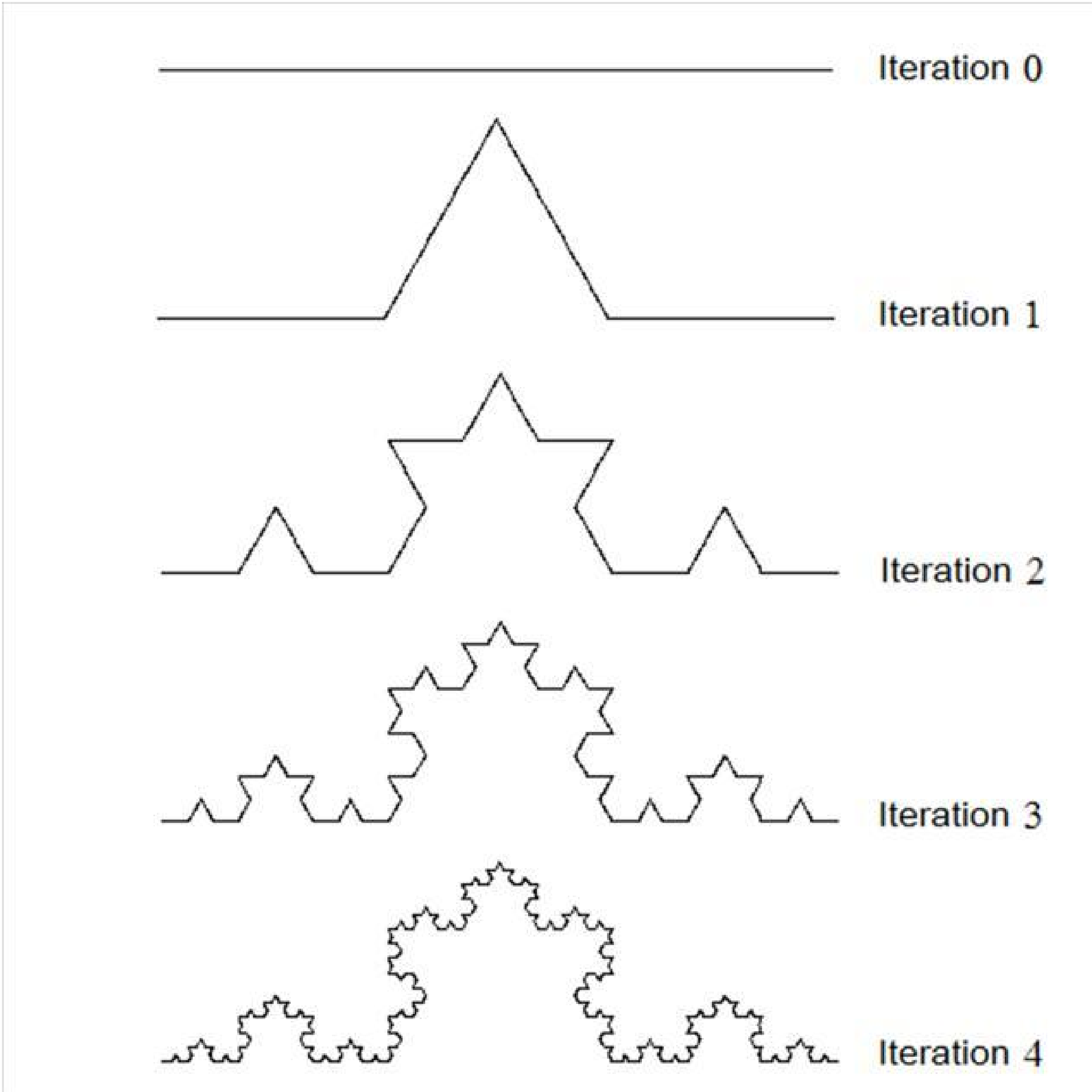

FIGURE 2. Four iterations of the von Koch curve.

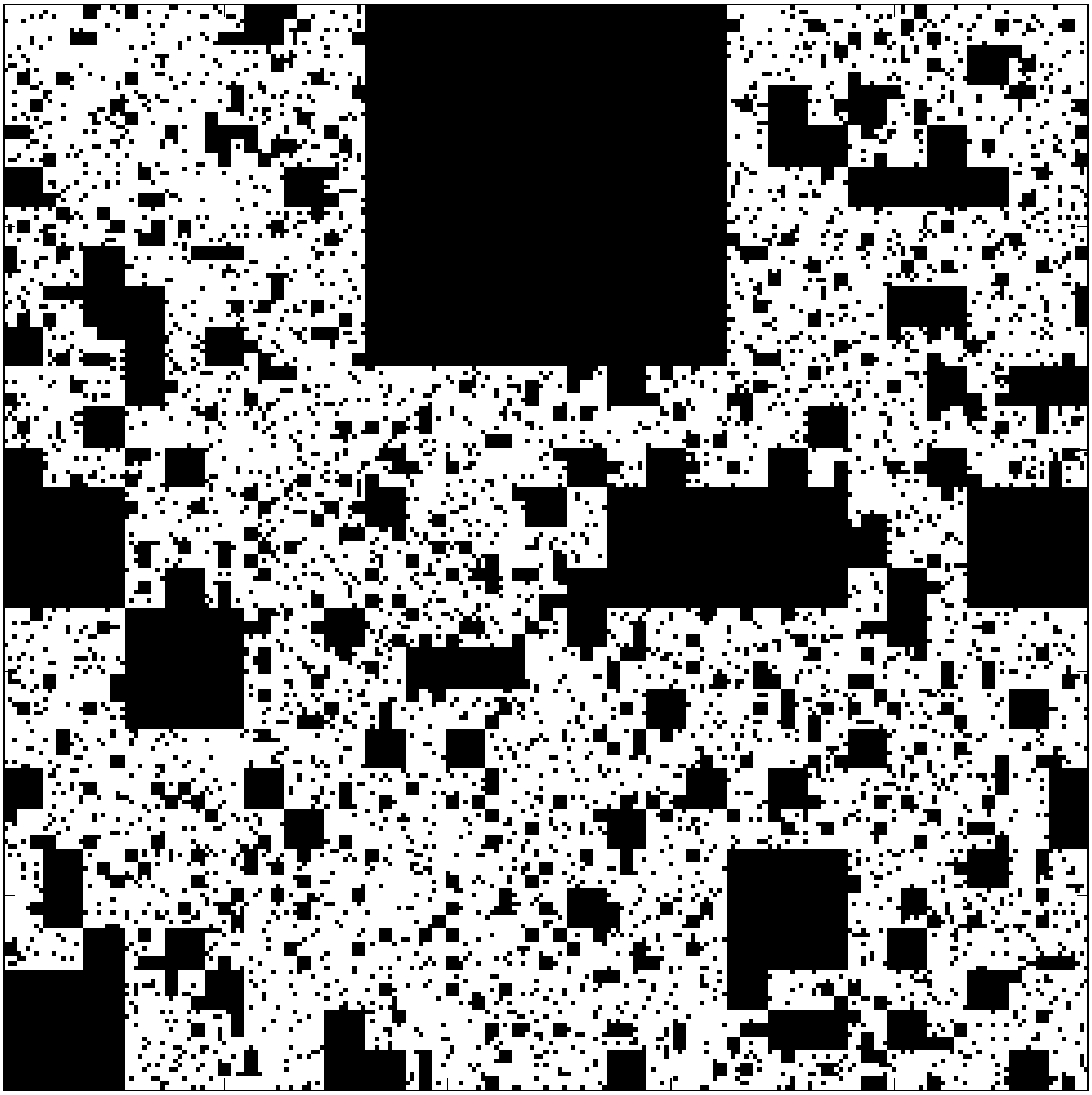

FIGURE 3. Non-self-similar Sierpinski carpet.