LAPLACE'S METHOD

We did perturbation methods for ODEs, however they are not the only types of asymptotic expansions. Consider the integral

$$f(x) = \int_x^\infty t^{-1} e^{x-t} dt; \qquad x \gg 1. \tag{1}$$

We solve this via by-parts: $u = t^{-1} \Rightarrow du = -t^{-2}$, and $dv = e^{x-t}dt \Rightarrow v = -e^{x-t}$,

$$f(x) = -\frac{1}{t}e^{x-t}\Big|_x^\infty - \int_x^\infty t^{-2}e^{x-t}dt = \frac{1}{x} - \int_x^\infty t^{-2}e^{x-t}dt.$$

If we keep doing by-parts we get

$$f(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \cdots + (-1)^{n-1}\frac{(n-1)!}{x^n} + R_n(x); \qquad R_n(x) = (-1)^n n! \int_x^\infty t^{-(n+1)}e^{x-t}dt. \tag{2}$$

Notice that $S_n(x) = f(x) - R_n(x)$ diverges as $n \to \infty$ for all $x \in \mathbb{R}$. This is because factorials grow faster than exponentials.

However, intuitively as $x \to \infty$ it seems like the approximation won't be too bad since it competes with n! From analysis we know

$$\left|\int_x^\infty t^{-(n+1)}e^{x-t}dt\right| \le \int_x^\infty \left|t^{-(n+1)}\right|\left|e^{x-t}\right|dt.$$

Since $t \ge x$,

$$\left|t^{-(n+1)}\right| \le x^{-(n+1)} \Rightarrow |R_n(x)| < n!x^{-(n+1)}\int_x^\infty e^{x-t}dt = n!x^{-(n+1)}.$$

So, as $x \to \infty$, $R_n(x) \to 0$ for fixed n, and even for small n such as $n = 3$, $R_n < 1/144 \approx 0.007$ for $x \ge 6$.

This is markedly different from Taylor series, which converges as $n \to \infty$. This shows us that convergence is not always necessary for a useful series.

Definition 1. Consider the series

$$A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \cdots + \frac{A_n}{x^n} + \frac{A_{n+1}}{x^{n+1}} + \cdots \equiv S_n(x) + \frac{A_{n+1}}{x^{n+1}} + \cdots. \tag{3}$$

The expression $S_n(x)$ is said to provide an asymptotic expansion of $f(x)$ as $x \to \infty$ if $\lim_{x\to\infty} x^n[f(x) - S_n(x)] = 0$ for a fixed n.

One of the most frequently used asymptotic methods in applied mathematics is Laplace's method, for integrals of the form

$$F(\lambda) = \int_\alpha^\beta g(t)e^{-\lambda f(t)}dt, \quad \lambda > 0. \tag{4}$$

For $\lambda \gg 1$, we transform the integral into a form that admits the dominant contribution from a finite portion of the path of integration. Suppose that portion is in a neighborhood of some t_0. Then we Taylor expand about t_0. For this class we will solve problems where $t_0 \in (\alpha, \beta)$ is the point at which $f(t)$ attains its absolute min; i.e., $f'(t_0) = 0$ and $f''(t_0) > 0$.

We can Taylor expand f as follows,

$$f(t) = f(t_0) + \cancelto{0}{f'(t_0)}(t - t_0) + \frac{1}{2}f''(t_0)(t-t_0)^2 + \cdots.$$

Then

$$e^{-\lambda f(t)} \approx e^{-\lambda f(t_0)}e^{-\frac{\lambda}{2}f''(t_0)(t-t_0)^2}.$$

Let

$$Q(t) \equiv e^{-\frac{\lambda}{2}f''(t_0)(t-t_0)^2}. \tag{5}$$

Notice that as $\frac{\lambda}{2}f''(t_0)$ gets bigger the slope drops precipitously. So, only the immediate neighborhood around t_0 matters because otherwise $Q \approx 0$. Hence for $\lambda \gg 1$, $g(t) \sim g(t_0)$. Then

$$F(\lambda) \equiv \int_\alpha^\beta g(t)e^{-\lambda f(t)}dt \sim g(t_0)e^{-\lambda f(t_0)}\int_\alpha^\beta Q(t)dt. \tag{6}$$

If we let $u = \sqrt{f''(t_0)/2}(t - t_0)$,

$$F(\lambda) \sim g(t_0)e^{-\lambda f(t_0)}\sqrt{\frac{2}{f''(t_0)}}\int_{-\infty}^{\infty} e^{-\lambda u^2}du = g(t_0)e^{-\lambda f(t_0)}\sqrt{\frac{2\pi}{\lambda f''(t_0)}} \tag{7}$$

as $x \to \infty$.

Now we may apply this to Stirling's approximation,

$$n! \sim \sqrt{2\pi n}n^n e^{-n}, \tag{8}$$

which is often used in Statistics.

We first need an integral representation of the factorial. This is in the form of the Gamma function

$$\Gamma(s) = \int_0^{\infty} x^{s-1}e^{-x}dx. \tag{9}$$

Notice that for $s = n \in \mathbb{N}$,

$$\Gamma(n) = \int_0^{\infty} x^{n-1}e^{-x}dx.$$

Let $u = x^{n-1} \Rightarrow du = (n-1)x^{n-2}$ and $dv = e^{-x}dx \Rightarrow v = -e^{-x}$, then

$$\Gamma(n) = (n-1)!$$

if we continue the integration by-parts. This shows that the formula is indeed the continuous analog of the factorial.

For Stirling's approximation, we are interested in $s \gg 1$. We need to first transform $\Gamma(s)$ into the form of Laplace's equation. Notice that

$$x^{s-1} = e^{(s-1)\ln x} \Rightarrow \Gamma(s) = e^{s\left[\ln x - \frac{1}{s}\ln x + \frac{x}{s}\right]},$$

so let $\eta(x) = \ln x - \frac{1}{s}\ln x + \frac{x}{s}$. Notice that $f'(x_0) = 0 \Rightarrow x_0 = s - 1$, which changes with s. To remedy this let us make a change of variables $t = x/(s-1)$. Then

$$\Gamma(s) = (s-1)^s \int_0^{\infty} e^{-(s-1)f(t)}dt; \qquad f(t) = t - \ln t.$$

Then we have $f'(t_0) = 0 \Rightarrow t_0 = 1$, then $f(t_0) = 1$. To write this in terms of the integral from Laplace's method, let $\lambda = s - 1$ and $f(t) = f(t_0) + [f(t) - f(t_0)]$,

$$\Gamma(\lambda) = \lambda^{\lambda+1}e^{-\lambda}\int_0^{\infty} e^{-\lambda[f(t)-f(t_0)]}dt. \tag{10}$$

Notice, however that we need $-\infty$ to ∞, so let $w = -\sqrt{f(t) - f(t_0)}$ for $t \leq t_0$ and $w = \sqrt{f(t) - f(t_0)}$ for $t \geq t_0$. Then,

$$w^2 = t - 1 - \ln t = \frac{1}{2}(t-1)^2 - \frac{1}{3}(t-1)^3 + \cdots, \tag{11}$$

but we need dt/dw, so let $t - 1 = a_0 w + o(w^2)$ and match terms gives us $a_0 = \sqrt{2}$. So,

$$\Gamma(\lambda) = \lambda^{\lambda+1}e^{-\lambda}\int_{-\infty}^{\infty} e^{-\lambda w^2}\frac{dt}{dw}dw \sim \lambda^{\lambda+1}e^{-\lambda}\int_{-\infty}^{\infty} e^{-\lambda w^2}\sqrt{2}dw = \lambda^{\lambda+1}e^{-\lambda}\sqrt{\frac{2\pi}{\lambda}} \tag{12}$$

Finally,

$$n! = \Gamma(\lambda = n) \sim n^{n+1}e^{-n}\sqrt{\frac{2\pi}{n}} = n^n e^{-n}\sqrt{2\pi n} \tag{13}$$