Spring 2014 solutions

(1) We find our formula for F first. We know for a spring $F = kx$, and since it takes 200N to stretch the spring .8m from its natural position, $F(.8) = \frac{4}{5}k = 200 \Rightarrow k = 250$. Then, $F = 250x$.

(a) $250x = 300 \Rightarrow x = \frac{300}{250} = \frac{6}{5} = 1.2$m

(b) $F(x) = 250x$N, so we get

$$W = 250\int_0^1 x\mathrm{d}x = 125x^2|_0^1 = 125\text{J}.$$

(2) We first find the derivative: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(1+e^x)^{-1/2}e^x$. Then, we plug this into the surface area formula,

$$\text{SA} = \int_a^b f(x)\sqrt{1+f'(x)^2}\mathrm{d}x = \int_0^1 \sqrt{1+e^x}\sqrt{1+\frac{e^{2x}}{4(1+e^x)}}\mathrm{d}x = \int_0^1 \sqrt{1+e^x+\frac{1}{4}e^{2x}}$$

$$= \int_0^1 \sqrt{(1+\frac{1}{2}e^x)^2}\mathrm{d}x = \int_0^1 (1+\frac{1}{2}e^x)\mathrm{d}x = x + \frac{e^x}{2}\bigg|_0^1 = 1 + \frac{e}{2} - \frac{1}{2} = \frac{1}{2}(1+e).$$

(3) Notice, this is easier if we convert the problem in terms of y, so $y = 3x^{2/3} \Rightarrow x = \left(\frac{y}{3}\right)^{3/2}$. Then we differentiate, $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{3}{2}\left(\frac{y}{3}\right)^{1/2}\frac{1}{3} = \frac{1}{2\sqrt{3}}\sqrt{y}$. Plugging this into the formula for arc length gives,

$$L = \int_a^b \sqrt{1+f'(y)^2}\mathrm{d}y = \int_3^{12}\sqrt{1+\frac{y}{12}}\mathrm{d}y$$

We solve this via u-sub, where $u = 1 + y/12 \Rightarrow \mathrm{d}u = \mathrm{d}y/12$, then

$$L = 12\int_{5/4}^2 \sqrt{u}\mathrm{d}u = 12\left(\frac{2}{3}\right)u^{3/2}\bigg|_{5/4}^2 = 8\left[2\sqrt{2} - \frac{5\sqrt{5}}{8}\right] = 16\sqrt{2} - 5\sqrt{5}.$$

(4) This is very similar to the cable problems we did and it is exactly like the sandbag problem in the homework. The starting weight is 105lb and the rate at which the water is escaping is 10lb every 200ft or 1/20lb/ft. Then the weight change of the system for any given change in distance Δx will be $F = 105 - \frac{1}{20}\Delta x$. Then the work is,

$$W = (105)(200) - \int_0^{200}\frac{x}{20}\mathrm{d}x = (105)(200) - \frac{1}{10}x^2\bigg|_0^{200}$$

$$= (105)(200) - \frac{200^2}{10} = (200)(105-20) = 1.7\text{x}10^4.$$

(5) We've done this type of problem plenty of times, so lets go straight to the volume of each infinitesimally small cylinder: $V = \pi r^2 h$ where $r = x$ and $h = \Delta y$, so $V = \pi x^2 \Delta y = \frac{\pi}{2} y \Delta y$, then the weight of each cylinder is: $F = 15\pi y \Delta y$. To pull each little piece up to $y = 2$ we exert a work of $W = 15\pi y(2 - y)\Delta y$. Now, to get the limits we notice that we move the first piece 1ft from $y = 1$ and the last piece 2ft from $y = 0$. Then, the work is

$$W = 15\pi \int_0^1 y(2 - y)\mathrm{d}y = 15\pi y^2\Big|_0^1 - 5\pi y^3\Big|_0^1 = 10\pi.$$

(6) Since we are using cylindrical shells we recall the formula for cylindrical shells,

$$V = 2\pi \int_a^b r(x)h(x)\mathrm{d}x.$$

(a) Here our radius will be $r = x$ and our height will be $h = y = 1/(1 + x^2)$, then we solve for our volume via u-sub

$$V = 2\pi \int_0^1 x\frac{1}{1 + x^2}\mathrm{d}x = \pi \int_1^2 \frac{\mathrm{d}u}{u} = \pi \ln u|_1^2 = \pi \ln 2.$$

(b) Here our radius will be $r = x + 1$ and our height remains the same, then

$$V = 2\pi \int_0^1 (x + 1)\frac{1}{1 + x^2}\mathrm{d}x = 2\pi \int_0^1 \frac{x\mathrm{d}x}{1 + x^2} + 2\pi \int_0^1 \frac{\mathrm{d}x}{1 + x^2}.$$

Now, the first integral we already solved in part a, and the second integral is our usual $\tan^{-1}$. Told ya he loves $\tan^{-1}$. This gives,

$$V = \pi \ln 2 + 2\pi \tan^{-1} x|_0^1 = \pi \ln 2 + \frac{\pi^2}{2}.$$

(7) Here it's a little tricky because our region is in the second quadrant, but all the theory is the same. We recall our formula for the washer method,

$$V = \pi \int_a^b (R(x)^2 - r(x)^2)\mathrm{d}x.$$

Here our range for x will be -1 to 0. Our big radius is $R = e$ and our little radius is $r = e^{-x}$, then

$$V = \pi \int_{-1}^0 (e^2 - e^{-2x})\mathrm{d}x = \pi e^2 x + \frac{\pi}{2} e^{-2x}\Big|_{-1}^0 = \frac{\pi}{2} + \pi e^2 - \frac{\pi}{2} e^2 = \frac{\pi}{2}(1 + e^2).$$

(8) For this problem its difficult to figure out what the region looks like, but the beauty of math is we don't have to see it to understand it. It's some region where the base goes from $y = 0$ to $y = \sec x$, and the height will be double the length of the base. Then our cross-sectional area is $A = y \cdot 2y = 2\sec^2 x$, and our volume is

$$V = 2 \int_0^{\pi/4} \sec^2 x\mathrm{d}x = 2 \tan x|_0^{\pi/4} = 2.$$