Spring 2014 solutions

(1) (a) $\lim_{n\to\infty} ne^{-n} = \lim_{n\to\infty} \frac{n}{e^n} = \lim_{n\to\infty} \frac{1}{e^n} = 0.$
Hence, it converges.

(b) $\lim_{n\to\infty} \sqrt{n^2+n} - n = \lim_{n\to\infty} \frac{n^2+n-n^2}{\sqrt{n^2+n}+n}$
$= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n}+1} = \frac{1}{2}.$
Hence, it converges.

(2) Notice $\Delta x = 1/2$, so $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$, $x_3 = 2.5$, $x_4 = 3$.

(a) Plugging this into the formula gives,

$$I_T = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)] = \frac{1}{4}\left[1 + \frac{2}{1.5} + \frac{2}{2} + \frac{2}{2.5} + \frac{1}{3}\right] = \frac{67}{60}.$$

(b) Plugging this into the formula gives,

$$I_S = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)] = \frac{1}{6}\left[1 + \frac{4}{1.5} + \frac{2}{2} + \frac{4}{2.5} + \frac{1}{3}\right] = \frac{11}{10}.$$

(3) (a) I find it easier to do a u-sub first, but you don't have to. Let $\xi = x/3 \Rightarrow 3\mathrm{d}\xi = \mathrm{d}x$, so $\int x\cosh(x/3)\mathrm{d}x = 9\int \xi\cosh\xi\mathrm{d}\xi$. We integrate this by parts using $u = \xi \Rightarrow \mathrm{d}u = \mathrm{d}\xi$ and $\mathrm{d}v = \cosh\xi\mathrm{d}\xi \Rightarrow v = \sinh\xi$. This gives,

$$I = 9\xi\sinh\xi - 9\int\sinh\xi\mathrm{d}\xi = 9\xi\sinh\xi - 9\cosh\xi + C = 3x\sinh\frac{x}{3} - 9\cosh\frac{x}{3} + C.$$

(b) We must first break this fraction up,

$$\frac{x^2+x}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} = \frac{A(x-1)^2 + B(x-1) + C}{(x-1)^3}$$
$$= \frac{Ax^2 - 2Ax + A + Bx - B + C}{(x-1)^3} = \frac{Ax^2 + (B-2A)x + A - B + C}{(x-1)^3}.$$

This gives us $A = 1 \Rightarrow B = 3 \Rightarrow C = 2$, then

$$I = \int\frac{\mathrm{d}x}{x-1} + 3\int\frac{\mathrm{d}x}{(x-1)^2} + 2\int\frac{\mathrm{d}x}{(x-1)^3} = \ln|x-1| - \frac{3}{x-1} - \frac{1}{(x-1)^2} + C.$$

(4) We use trig-sub with $x = 2\tan\theta \Rightarrow \mathrm{d}x = 2\sec^2\theta$,

$$I = \int \frac{8\tan^3\theta 2\sec^2\theta}{2\sec\theta}\mathrm{d}\theta = 8\int \tan^3\theta\sec\theta\mathrm{d}\theta = 8\int \tan\theta(\sec^2\theta - 1)\sec\theta\mathrm{d}\theta$$

We solve this via u-sub with $u = \sec\theta \Rightarrow \mathrm{d}u = \sec\theta\tan\theta$,

$$8\int(u^2-1)\mathrm{d}u = 8\frac{u^3}{3} - u + C = \frac{8}{3}\sec^3\theta - 8\sec\theta + C.$$

Now, we must put it back into x. Notice $\tan\theta = x/2$, so the hypotenuse is $\sqrt{x^2+4}$, then $\sec\theta = \sqrt{x^2+4}/2$, then

$$I = \frac{1}{3}\left(\sqrt{x^2+4}\right)^3 - 4\sqrt{x^2+4} + C.$$

(5) We break this up via partial fractions,

$$\frac{5x^3 - 3x^2 + 2x - 1}{x^4 + x^2} = \frac{5x^3 - 3x^2 + 2x - 1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}.$$

Finding the common denominator and equating the numerators gives,

$$\begin{aligned} 5x^3 - 3x^2 + 2x - 1 &= Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2 \\ &= Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2 \\ &= (A+C)x^3 + (B+D)x^2 + Ax + B. \end{aligned}$$

We get $A = 2$ and $B = -1$ straight away, then $C = 3$ and $D = -2$, then

$$\begin{aligned} I &= 2\int\frac{\mathrm{d}x}{x} - \int\frac{\mathrm{d}x}{x^2} + \int\frac{3x-2}{x^2+1}\mathrm{d}x \\ &= 2\int\frac{\mathrm{d}x}{x} - \int\frac{\mathrm{d}x}{x^2} + 3\int\frac{x}{x^2+1}\mathrm{d}x - 2\int\frac{\mathrm{d}x}{x^2+1} \\ &= 2\ln|x| + \frac{1}{x} + \frac{3}{2}\ln|x^2+1| - 2\tan^{-1}x + C. \end{aligned}$$

(6) We solve this via trig-sub with $x = \frac{1}{2}\sin\theta \Rightarrow \mathrm{d}x = \frac{1}{2}\cos\theta \mathrm{d}\theta$,

$$I = \int \frac{\frac{1}{8}\sin^3\theta \frac{1}{2}\cos\theta}{\cos^3\theta}\mathrm{d}\theta = \frac{1}{16}\int \frac{\sin\theta(1-\cos^2\theta)}{\cos^2\theta}\mathrm{d}\theta$$
$$= \frac{1}{16}\left[\int \sec\theta\tan\theta\mathrm{d}\theta - \int \sin\theta\mathrm{d}\theta\right] = \frac{1}{16}[\sec\theta + \cos\theta + C].$$

Now we put this back in terms of x, noting that $\sin\theta = 2x$, so the adjacent side is $\sqrt{1-4x^2}$, then

$$\frac{1}{16}\left[\frac{1}{\sqrt{1-4x^2}} + \sqrt{1-4x^2} + C\right].$$

(7) We solve this via integration by parts, with $u = \ln(x^3 - x)$ $\Rightarrow \mathrm{d}u = \frac{3x^2-1}{x^3-x}\mathrm{d}x$ and $\mathrm{d}v = \mathrm{d}x \Rightarrow v = x$,

$$I = x\ln(x^3-x) - \int \frac{3x^2-1}{x^3-x}x\mathrm{d}x = x\ln(x^3-x) - \int \frac{3x^2-1}{x^2-1}\mathrm{d}x.$$

Now, we must break up the second integral via long division and partial fractions,

$$\frac{3x^2-1}{x^2-1} = 3 + \frac{2}{x^2-1} = 3 + \frac{2}{(x-1)(x+1)} = 3 + \frac{A}{x-1} + \frac{B}{x+1}.$$

Then, finding the common denominator and matching the numerators, gives

$$2 = A(x+1) + B(x-1) = Ax + A + Bx - B = (A+B)x + A - B.$$

This gives, $A = -B = 1$, then

$$I = x\ln(x^3-x) - \int 3\mathrm{d}x - \int \frac{\mathrm{d}x}{x-1} + \int \frac{\mathrm{d}x}{x+1} = x\ln(x^3-x) - 3x - \ln|x-1| + \ln|x+1| + C.$$

(8) (a) It diverges because,

$$\int_0^{\pi/2} \tan^2 x\mathrm{d}x = \lim_{t\to\frac{\pi}{2}^-} \int_0^t \sec^2 x\mathrm{d}x - \int_0^t \mathrm{d}x = \lim_{t\to\frac{\pi}{2}^-} (\tan x - x)|_0^t = \lim_{t\to\frac{\pi}{2}^-} \tan t - t = \infty$$

(b) We evaluate this equation via u-sub with $u = \tan^{-1} x \Rightarrow (1/(1+x^2))\mathrm{d}x$.

$$\int_0^\infty \frac{\tan^{-1}x}{1+x^2}\mathrm{d}x = \lim_{t\to\infty}\int_0^t \frac{\tan^{-1}x}{1+x^2}\mathrm{d}x = \lim_{t\to\infty}\int_0^{\tan^{-1}t} u\mathrm{d}u$$
$$= \lim_{t\to\infty}\frac{1}{2}u^2\Big|_0^{\tan^{-1}t} = \lim_{t\to\infty}\frac{1}{2}(\tan^{-1}t)^2 = \frac{\pi^2}{8}.$$

(9) (a) We can just take the highest term in the numerator, which is $\sqrt{x}$ and the highest term in the denominator, which is x^2, then $\sqrt{x}/x^2 = x^{-3/2}$. We use the limit comparison test because it's tough to come up with a good direct comparison, so

$$\lim_{x\to\infty} \frac{\sqrt{x+2}/x^2}{x^{-3/2}} = \lim_{x\to\infty} \frac{\sqrt{x+2}}{\sqrt{x}} = \lim_{x\to\infty} \sqrt{1+\frac{2}{x}} = 1.$$

This shows that the comparison is valid, so since $\int_1^\infty \frac{\mathrm{d}x}{x^{3/2}}$ converges because $p > 1$, $\int_1^\infty \frac{\sqrt{x+2}}{x^2}\mathrm{d}x$ also converges by the limit comparison test.

(b) It's a bit more difficult to see the comparison here, but we know $1/e^x$ goes to 0 very fast, so our hunch is that it converges. Notice that in our integral the denominator has a positive term added to e^x on $[1,\infty)$, so $\frac{1}{e^x+\sqrt{x}} < \frac{1}{e^x}$ on $[1,\infty)$. Now,

$$\int_1^\infty \frac{1}{e^x}\mathrm{d}x = \lim_{t\to\infty} \int_1^t e^{-x} = \lim_{t\to\infty} -e^{-x}\big|_1^t = \lim_{t\to\infty} -e^{-t} + \frac{1}{e} = \frac{1}{e}$$

So it converges, therefore by the direct comparison test $\int_1^\infty \frac{\mathrm{d}x}{e^x+\sqrt{x}}$ also converges.