Spring 2014 solutions

(1) (a) We use the limit comparison test. To find what comparison we need we take the biggest term in the numerate and denominator i.e. $n/n^3 = 1/n^2$. Now, we take the limit to show this is a valid comparison,

$$\lim_{n\to\infty} \frac{(n-\ln(n))/(2n^3+1)}{1/n^2} = \lim_{n\to\infty} \frac{n^3 - n^2 \ln n}{2n^3+1} = \lim_{n\to\infty} \frac{1-\ln(n)/n}{2+1/n} = \frac{1}{2}.$$

Now, since $\sum_{n=1}^{\infty} 1/n^2$ converges because $p > 1$, the original series converges by the limit comparison test.

(b) We use limit comparison again. Here lets use $1/n$ and take the limit,

$$\lim_{n\to\infty} \frac{(n^2+1)/(n^3+n^2)}{1/n} = \lim_{n\to\infty} \frac{n^3+n}{n^3+n^2} = \lim_{n\to\infty} \frac{1+1/n^2}{1+1/n} = 1.$$

Since $\sum_{n=1}^{\infty} 1/n$ diverges because $p = 1$, the original series also diverges by the limit comparison test.

(2) (a) We have a feeling this diverges, so lets take the limit of the "n^{th}" term $\lim_{n\to\infty} \cos\frac{1}{n} = 1 \neq 0$, and hence the sum diverges.

(b) It's best to first massage the series into a form we are comfortable with, $\sum_{n=1}^{\infty} \frac{4^{n+1}}{n^{n/2}} = 4\sum_{n=1}^{\infty} \frac{4^n}{\left(n^{1/2}\right)^n}$. Now, lets use the root test,

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{4}{\sqrt{n}} = 0 < 1.$$

Therefore, the series converges by the root test.

(3) (a) We use integral test for this. Since we should be experts in u-sub I wont show those steps,

$$\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}} \mathrm{d}x = 2\int_1^{\infty} e^{-u} \mathrm{d}u = -2e^{-u}\Big|_1^{\infty} = \frac{2}{e}.$$

By the integral test, since the integral converges, so does the original series.

(b) There are a few ways of doing this problem, but lets take the suggestion used in class because that's the most direct way of doing this problem. Lets use limit comparison by taking the highest term in the denominator and the numerator which will be $n^n/(2n)^n = 1/2^n$. Now lets take the limit,

$$\lim_{n\to\infty} \frac{(2^n + n^n)/(1 + (2n)^n)}{1/2^n} = \lim_{n\to\infty} \frac{4^n + (2n)^n}{1 + (2n)^n} = \lim_{n\to\infty} \frac{(2/n)^n + 1}{1/(2n)^n + 1} = 1.$$

Since $\sum_{n=1}^{\infty} 1/2^n$ converges by the geometric series because $1/2 < 1$, by the limit comparison test the original series also converges.

(4) (a) We have an alternating series, but lets first test for absolute convergence via ratio test,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{8^{n+1}}{(n+1)!} \cdot \lim_{n\to\infty} \frac{n!}{8^n}\right| = \lim_{n\to\infty} \frac{8}{n+1} = 0 < 1.$$

Therefore, by the ratio test, the series is absolutely convergent.

(b) To avoid any deductions we really should use limit comparison. We can use direct comparison as well, but that gets really tricky. Fortunately, I think the grader was lenient, probably more lenient than I would have been.

We take the largest term in the numerator and denominator to get $n/\sqrt{n^3} = 1/\sqrt{n}$, now lets take the limit,

$$\lim_{n\to\infty} \frac{n/\sqrt{n^3+1}}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n^3}}{\sqrt{n^3+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1 + 1/n^3}} = 1.$$

Since $\sum_{n=1}^{\infty}$ diverges by p-series because p¡1, the original series can not be absolutely convergent. Since it's an alternating series, lets try to prove conditional convergence.

First we show that the n^{th} term goes to zero,

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^3+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{n + 1/n^2}} = 0.$$

Now to show it's decreasing. We must take the derivative for this. Again, the grader was very generous, but I certainly would have marked it incorrect if the derivative wasn't taken.

$$\left(\frac{n}{\sqrt{n^3+1}}\right)' = \frac{2 - n^3}{2(n^3+1)^{3/2}}.$$

Notice, this is < 0 for $n \geq 2$, and therefore by the alternating series test, the series converges conditionally.

(5) We go straight to ratio test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(x+2)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{2^n(x+2)^n}\right| = \lim_{n\to\infty} 2\frac{\sqrt{n}}{\sqrt{n+1}}|x+2|$$
$$= \lim_{n\to\infty} 2\frac{1}{\sqrt{1+1/n}}|x+2| = 2|x+2| < 1 \Rightarrow |x+2| < 1/2.$$

Hence, the radius of convergence is $R = 1/2$ and the series converges absolutely for $-2.5 < x < -1.5$. Now, we must test the end points. When $x = -1.5$, our series becomes $\sum_{n=1}^{\infty} 7^n/\sqrt{n}$, but $\lim_{n\to\infty} 7^n/\sqrt{n} = \infty \neq 0$, so it diverges there. For $x = -2.5$, our series becomes $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$. Taking the limit gives, $\lim_{n\to\infty} 1/\sqrt{n} = 0$. Further, $1/\sqrt{n} > 1/\sqrt{n+1}$. Therefore it converges at this point by the alternating series test. Hence, our interval of convergence is $-2.5 \leq x < -1.5$.

(6) Again we use ratio test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\sqrt{n+1}(x-1)^{n+1}}{(n+2)!}\cdot\frac{(n+1)!}{\sqrt{n}(x-1)^n}\right| = \lim_{n\to\infty}\frac{\sqrt{n+1}}{(n+2)\sqrt{n}}|x-1|$$
$$= \lim_{n\to\infty}\frac{\sqrt{1+1/n}}{(n+2)}|x-1| = 0.$$

Hence, this has a radius of convergence $R = \infty$ and interval of convergence $(-\infty, \infty)$.

(7) (a) We know the Taylor series of $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$, hence

$$e^{-x^3} = \sum_{n=0}^{\infty}\frac{(-x^3)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{3n}}{n!}$$

However, if you used this it's all or nothing, so you should have only used this if you were 100% confident.

(b) Here we simply integrate term by term and plug in our values,

$$\int_0^{0.1} f(x)\mathrm{d}x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+1}}{(3n+1)n!}\bigg|_{x=0.1} = \sum_{n=0}^{\infty}\frac{(-1)^n(.1)^{3n+1}}{(3n+1)n!}.$$

(c) For this we need the alternating series error, which is just the next term, $|R_1| \leq |x^7/2| \leq (.1)^7/2$.

(8) (a) For this problem we are forced to compute the Taylor series manually, $f(1) = e$, $f'(1) = e^x + xe^x\big|_{x=1} = 2e$, $f''(1) = 2e^x + xe^x\big|_{x=1} = 3e$, $f'''(1) = 3e^x + xe^x\big|_{x=1} = 4e$, therefore

$$xe^x \approx e + 2e(x-1) + \frac{3e}{2}(x-1)^2 + \frac{2e}{3}(x-1)^3.$$

(b) We need to compute the fourth derivative for our Taylor remainder, $f^{(4)}(x) = 4e^x + xe^x$, and we need to bound this on our interval. Plugging in 2 provides such a bound, $|f^{(4)}(x)| \leq 6e^2$, so we choose $M = 6e^2$. Then,

$$|R_3| \leq \left|\frac{6e^2}{4!}(x-1)^4\right| \leq \frac{6e^2}{24} = \frac{e^2}{4}.$$