Spring 2011 solutions

(1) We use disks to solve this,

$$V = \pi \int_0^1 (xe^x)^2 \mathrm{d}x = \pi \int_0^1 x^2 e^{2x} \mathrm{d}x.$$

We solve this via integration by parts with $u = x^2 \Rightarrow \mathrm{d}u = 2x\mathrm{d}x$ and $\mathrm{d}v = e^{2x}\mathrm{d}x \Rightarrow v = e^{2x}/2$,

$$V = \frac{\pi}{2} x^2 e^{2x}\Big|_0^1 - \pi \int_0^1 xe^{2x}\mathrm{d}x.$$

This is another integration by parts with $u = x \Rightarrow \mathrm{d}u = \mathrm{d}x$ and $\mathrm{d}v = e^{2x}\mathrm{d}x \Rightarrow v = e^{2x}/2$,

$$V = \frac{\pi e^2}{2} - \frac{\pi}{2} xe^{2x}\Big|_0^1 + \pi \int_0^1 \frac{1}{2} e^{2x}\mathrm{d}x = \frac{\pi e^2}{2} - \frac{\pi e^2}{2} + \frac{\pi}{4} e^{2x}\Big|_0^1 = \frac{\pi}{4}(e^2 - 1).$$

(2) (a) This is a typical partial fractions problem,

$$\frac{6x+8}{x(x+2)^2} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2}.$$

This gives us $A(x+2)^2 + Bx(x+2) + Cx = A(x^2+4x+4) + B(x^2+2x) + Cx = (A+B)x^2 + (4A+2B+C)x + 4A = 6x+8$. The easiest thing to solve for is $A = 2 \Rightarrow B = -2$, plugging these into the middle term gives, $C = -4$. Now we put these into the integra,

$$\int \frac{6x+8}{x(x+2)^2}\mathrm{d}x = \int \left(\frac{2}{x} - \frac{2}{x+2} + \frac{2}{(x+2)^2}\right)\mathrm{d}x = 2\ln|x| - 2\ln|x+2| - \frac{2}{x+2} + C.$$

(b) We solve this via integration by parts with $u = x \Rightarrow \mathrm{d}u = \mathrm{d}x$ and $\mathrm{d}v = \sec^2 x\mathrm{d}x \Rightarrow v = \tan x$,

$$\int x\sec^2 x\mathrm{d}x = x\tan x - \int \tan x\mathrm{d}x = x\tan x + \ln|\cos x| + C.$$

Recall, we solve $\int \tan x\mathrm{d}x$ by breaking it up into sin and cos and using u-sub.

(3) (a) This is another partial fractions problem,

$$\frac{x^2+2x+3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

From this we get $Ax^2+A+Bx^2+Cx = (A+B)x^2+Cx+A = x^2+2x+3$. Then we solve for the coefficients $A=3, C=2 \Rightarrow B=-2$, and integrate

$$\int \frac{x^2+2x+3}{x(x^2+1)}\mathrm{d}x = \int \left(\frac{3}{x} + \frac{2}{1+x^2} - \frac{2x}{x^2+1}\right)\mathrm{d}x = 3\ln|x| + 2\tan^{-1}x - \ln|x^2+1| + C.$$

(b) This is a typical u-sub problem with $u=\sqrt{x} \Rightarrow \mathrm{d}u = 1/2\sqrt{x}\mathrm{d}x$,

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\mathrm{d}x = 2\int \cos u\mathrm{d}u = 2\sin\sqrt{x} + C.$$

(4) (a) This is a trig integral problem where we convert $\sin^2 x$,

$$I = \int \sin^3 x\cos^2 x\mathrm{d}x = \int (1-\cos^2 x)\cos^2 x\sin x\mathrm{d}x.$$

Now, we use u-sub with $u=\cos x \Rightarrow \mathrm{d}u = -\sin x\mathrm{d}x$,

$$I = \int (u^4-u^2)\mathrm{d}u = \frac{1}{5}u^5 - \frac{1}{3}u^3 = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C.$$

(b) This is a trig-sub problem where $x=\sin\theta \Rightarrow \mathrm{d}x = \cos\theta\mathrm{d}\theta$,

$$\begin{aligned}\int \frac{x^2\mathrm{d}x}{\sqrt{1-x^2}} &= \int \frac{\sin^2\theta\cos\theta}{\sqrt{1-\sin^2\theta}}\mathrm{d}\theta = \int \frac{\sin^2\theta\cos\theta}{\cos\theta}\mathrm{d}\theta = \int \sin^2\theta\mathrm{d}\theta = \int \frac{1}{2}(1-\cos 2\theta)\mathrm{d}\theta \\ &= \frac{\theta}{2} - \frac{1}{4}\sin 2\theta + C = \frac{1}{2}\sin^{-1}x - \frac{1}{2}\sin\theta\cos\theta + C = \frac{1}{2}\sin^{-1}x - \frac{1}{2}x\sqrt{1-x^2} + C.\end{aligned}$$

(5) The next two are improper integral problems.

(a) Here we first take the limit and then apply our u-sub of $u = \tan^{-x} \Rightarrow \mathrm{d}u = \frac{\mathrm{d}x}{1+x^2}$,

$$\int_0^\infty \frac{\tan^{-1}x}{1+x^2}\mathrm{d}x = \lim_{t\to\infty}\int_0^t \frac{\tan^{-1}x}{1+x^2}\mathrm{d}x = \lim_{t\to\infty}\int_0^{\tan^{-1}t} u\mathrm{d}u = \lim_{t\to\infty}\frac{u^2}{2}\Bigg|_0^{\tan^{-1}t} = \lim_{t\to\infty}\frac{1}{2}(\tan^{-1}t)^2 = \frac{\pi^2}{8}.$$

(b) Again, we first include the limit then we use "by parts" using $u = \ln x \Rightarrow \mathrm{d}u = \mathrm{d}x/x$ and $\mathrm{d}v = x^2\mathrm{d}x \Rightarrow v = x^3/3$,

$$\int_0^2 x^2 \ln x \mathrm{d}x = \lim_{t\to 0}\int_t^2 x^2 \ln x \mathrm{d}x = \lim_{t\to 0}\left[\frac{1}{3}x^3 \ln x\bigg|_t^2 - \int_t^2 \frac{1}{3}x^2\mathrm{d}x\right] = \lim_{t\to 0}\left[\frac{8}{3}\ln 2 - \frac{1}{3}t^3\ln t - \frac{1}{9}x^3\right]_t^2$$
$$= \lim_{t\to 0}\left[\frac{8}{3}\ln 2 - \frac{8}{9} + \frac{t^3}{9} - \frac{1}{3}t^3 \ln t\right] = \frac{8}{3}\ln 2 - \frac{8}{9}.$$

We get this by employing

$$\lim_{t\to 0} t^3 \ln t = \lim_{t\to 0}\frac{\ln t}{t^{-3}} = \lim_{t\to 0}\frac{1}{-3t^{-3}} = 0.$$

(6) Remember the standard forms of these series!

(a) We go straight to ratio test,

$$\lim_{t\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{t\to\infty}\left|\frac{(n+1)^2}{(2n+2)!}\cdot\frac{(2n)!}{n^2}\right| = \lim_{t\to\infty}\left|\frac{(1+1/n)^2}{(2n+2)(2n+1)}\right| = 0 < 1.$$

Hence, it converges.

(b) This looks like it diverges pretty badly so we just take the limit of the "n^{th}" term,

$$\lim_{t\to\infty} 2^{1/n} = 1 \neq 0.$$

Hence, it diverges.

(c) The easiest thing to do here is use limit comparison,

$$\lim_{t\to\infty}\frac{(1+3^n)/(1+4^n)}{3^n/4^n} = \lim_{t\to\infty}\frac{1+3^n}{1+4^n}\cdot\frac{4^n}{3^n} = \lim_{t\to\infty}\frac{1+12^n}{1+12^n} = 1.$$

So, this is a valid comparison. Since $\sum_{n=1}^{\infty}(3/4)^n$ converges by the geometric series because $3/4 < 1$, therefore $\sum_{n=1}^{\infty}\frac{1+3^n}{1+4^n}$ converges by the limit comparison test.

(d) We can do this one by direct comparison, but if you're not sure you should just use limit comparison. Notice that $\frac{\sqrt{n-1}}{3n^2+4} \leq \frac{\sqrt{n}}{3n^2} = \frac{1}{3n^{3/2}}$. Since $\frac{1}{3}\sum_{n=1}^{\infty} 1/n^{3/2}$ converges by p-series because $p > 1$, $\sum_{n=1}^{\infty}\frac{\sqrt{n-1}}{3n^2+4}$ converges by the direct comparison test.

(7) As per usual we first employ the ratio test,

$$\lim_{t\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{t\to\infty}\left|\frac{(x-2)^{n+1}}{\sqrt{n+2}}\cdot\frac{\sqrt{n+1}}{(x-2)^n}\right| = \lim_{t\to\infty}\sqrt{\frac{n+1}{n+2}}|x-2| = \lim_{t\to\infty}\sqrt{\frac{1+1/n}{1+2/n}}|x-2| = |x-2|.$$

Since we need $|x-2|<1$ by the ratio test, the radius of convergence is $R=1$, and the interval of absolute convergence is $1<x<3$. Now we must test the end points. When $x=1$, our series becomes $\sum_{n=1}^{\infty} 1/\sqrt{n+1} = \sum_{n=2}^{\infty} 1/\sqrt{n}$ diverges by p-series because $p<1$. When $x=3$, our series becomes $\sum_{n=1}^{\infty}(-1)^n/\sqrt{n+1}$. We first take the limit of the n^{th} term, $\lim_{t\to\infty} 1/\sqrt{n+1}=0$. Next we show that it's decreasing, $1/\sqrt{n+1} > 1/\sqrt{n+2}$. Therefore, by the alternating series test, it converges. So, our interval of convergence is $1<x\le 3$.

(8) Notice that $f(\pi/4)=\sqrt{2}/2$, $f'(\pi/4)=-\sqrt{2}/2$, $f''(\pi/4)=-\sqrt{2}/2$ and hence

$$f(x)\approx \frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)-\frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2.$$

(9) Recall that the series for exponentials about $x=0$ is $e^x=\sum_{n=0}^{\infty} x^n/n!$.

(a) Now we just plug in $-x^5$ and multiply out by x,

$$xe^{-x^5} = x\sum_{n=0}^{\infty}\frac{(-x^5)^n}{n!} = x\sum_{n=0}^{\infty}\frac{(-1)^n x^{5n}}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{5n+1}}{n!}.$$

(b) Now we integrate term by term,

$$\int_0^{0.1} xe^{-x^5}\,\mathrm{d}x = \int_0^{0.1}\sum_{n=0}^{\infty}\frac{(-1)^n x^{5n+1}}{n!} = \sum_{n=0}^{\infty}\int_0^{0.1}\frac{(-1)^n x^{5n+1}}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{5n+2}}{(5n+2)n!}\Bigg|_0^{0.1}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n(0.1)^{5n+2}}{(5n+2)n!} = \sum_{n=0}^{\infty}\frac{(-1)^n(10)^{-(5n+2)}}{(5n+2)n!}$$

(c) We see here that the exponential gets large very quickly and that it's an alternating series. The error for an alternating series is just the next term from the truncation, so lets just compute each term and find where it gives us the desired error and then just take all the terms up to but not including that term,

$$n=0:\frac{1}{200}\qquad n=1:-\frac{10^{-7}}{7}\qquad n=2:\frac{10^{-12}}{24}<10^{-8}.$$

Therefore the following is correct up to 10^{-8},

$$\int_0^{0.1} xe^{-x^5}\,\mathrm{d}x \approx \frac{1}{200}-\frac{10^{-7}}{7}.$$

(10) We have to find the points at which these two curves intersect, $2\cos\theta = 1 \Rightarrow \theta = \pi/3, -\pi/3$. Notice that we want the interval $-\pi/3 \le \theta \le \pi/3$ because as hypothesized by the problem, $2\cos\theta$ is larger in that region. Now we just plug into our formula and integrate,

$$\frac{1}{2}\int_{-\pi/3}^{\pi/3}(4\cos^2\theta - 1)\mathrm{d}\theta = \int_{-\pi/3}^{\pi/3}(1+\cos 2\theta)\mathrm{d}\theta - \frac{\theta}{2}\bigg|_{-\pi/3}^{\pi/3} = \theta + \frac{1}{2}\sin 2\theta - \frac{\theta}{2}\bigg|_{-\pi/3}^{\pi/3} = \frac{\sqrt{3}}{2}.$$

(11) Lets first find the respective derivatives,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{1+t} \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1+t-t}{(1+t)^2} = \frac{1}{(1+t)^2}.$$

Therefore, $\mathrm{d}y/\mathrm{d}x = 1+t$, and in the same vein $\mathrm{d}^2y/\mathrm{d}x^2 = \mathrm{d}y'/\mathrm{d}x = \frac{\mathrm{d}y'/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = (1+t)^2$.

Fall 2011 solutions

(1) First lets calculate the respective derivatives, $\mathrm{d}x/\mathrm{d}t = -2\cos t \sin t = -\sin 2t$ and $\mathrm{d}y/\mathrm{d}t = 2\sin t\cos t = \sin 2t$.

(a) We plug into our arc length formula,

$$L = \int_0^{\pi/4} \sqrt{2\sin^2 2t}\mathrm{d}t = \sqrt{2}\int_0^{\pi/4} \sin 2t\mathrm{d}t = \frac{-\sqrt{2}}{2}\cos 2t|_0^{\pi/4} = \frac{\sqrt{2}}{2}.$$

(b) We plug into the surface area formula,

$$\mathrm{SA} = \int_0^{\pi/4} 2\pi \sin^2 t\sqrt{2}\sin 2t\mathrm{d}t = \pi\sqrt{2}\int_0^{\pi/4}(1-\cos 2t)\sin 2t\mathrm{d}t = \pi\sqrt{2}\int_0^{\pi/4}(\sin 2t - \cos 2t\sin 2t)\mathrm{d}t$$
$$= \pi\sqrt{2}\left[\frac{-1}{2}\cos 2t|_0^{\pi/4} - \int_0^{\pi/4}\frac{1}{2}\sin 4t\mathrm{d}t\right] = \frac{\sqrt{2}}{2}\pi + \frac{\pi\sqrt{2}}{8}\cos 4t|_0^{\pi/4} = \frac{\sqrt{2}}{2}\pi - \frac{\sqrt{2}}{4}\pi = \frac{\sqrt{2}}{4}\pi.$$

(2) Lets convert this to cartesian coordinates $x = r\cos\theta = 4\sin\theta\cos\theta = 2\sin 2\theta$ and $y = r\sin\theta = 4\sin^2\theta$.

(a) Now lets find the respective derivatives, $\mathrm{d}y/\mathrm{d}\theta = 8\sin\theta\cos\theta = 4\sin 2\theta$ and $\mathrm{d}x/\mathrm{d}\theta = 4\cos 2\theta$, then we get

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\theta=\pi/3} = \left.\frac{\sin 2\theta}{\cos 2\theta}\right|_{\theta=\pi/3} = -\sqrt{3}.$$

(b) To find the area we notice that the bounds will be $\theta = 0$ and $\theta = \pi$,

$$A = \frac{1}{2}\int_0^{\pi} 16\sin^2\theta\mathrm{d}\theta = 4\int_0^{\pi}(1-\cos 2\theta)\mathrm{d}\theta = 4\theta - 2\sin\theta|_0^{\pi} = 4\pi.$$

(3) (a) This is a typical partial fractions problem,

$$\frac{4x+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

This gives, $A(x+1)^2 + Bx(x+1) + Cx = A(x^2+2x+1) + B(x^2+x) + Cx = (A+B)x^2 + (2A+B+C)x + A = 4x+1$, which gives $A = 1, B = -1, C = 3$. Then the integral becomes,

$$\int \frac{4x+1}{x(x+1)^2}\mathrm{d}x = \int\left(\frac{1}{x} - \frac{1}{x+1} + \frac{3}{(x+1)^2}\right)\mathrm{d}x = \ln|x| - \ln|x+1| - \frac{3}{x+1} + C.$$

(b) This is a typical trig-sub problem, where $x = 2\sin\theta \Rightarrow \mathrm{d}x = 2\cos\theta \mathrm{d}\theta$,

$$\int \frac{\mathrm{d}x}{(4-x)^{3/2}} = \int \frac{2\cos\theta \mathrm{d}\theta}{(4-4\sin^2\theta)^{3/2}} = \int \frac{2\cos\theta}{(2\cos\theta)^3}\mathrm{d}\theta = \int \frac{\mathrm{d}\theta}{4\cos^2\theta}$$
$$= \frac{1}{4}\int \sec^2\theta \mathrm{d}\theta = \frac{1}{4}\tan\theta = \frac{x}{4\sqrt{4-x^2}} + C.$$

(4) (a) This is a typical partial fractions problem, but we already did the partial fractions in 3a from Spring 2011 so we go straight to the coefficients: $(A+B)x^2 + Cx + A = 3x - 1$, so we get $A = -1, B = 1, C = 3$. Now we integrate,

$$\int \frac{3x-1}{x(x^2+1)}\mathrm{d}x = \int \left(\frac{x}{x^2+1} + \frac{3}{x^2+1} - \frac{1}{x}\right)\mathrm{d}x = \frac{1}{2}\ln|x^2+1| + 3\tan^{-1}x - \ln|x| + C.$$

(b) We use by parts with $u = \ln x \Rightarrow \mathrm{d}u = \mathrm{d}x/x$ and $\mathrm{d}v = x^{-1/2}\mathrm{d}x \Rightarrow v = 2x^{1/2}$,

$$\int \frac{\ln x}{\sqrt{x}}\mathrm{d}x = \int x^{-1/2}\ln x \mathrm{d}x = 2\sqrt{2}\ln x - 2\int \frac{\mathrm{d}x}{\sqrt{x}} = 2\sqrt{x}\ln x - 4\sqrt{x} + C.$$

Notice that I did not include absolute values here, because absolute values would make it incorrect.

(5) Both of these are improper integrals.

(a) We already did the u-sub in problem 3b Spring 2011, we will skip that step,

$$\int_0^1 \frac{\cos\sqrt{x}}{\sqrt{x}}\mathrm{d}x = \lim_{t\to 0}\int_t^1 \frac{\cos\sqrt{x}}{\sqrt{x}}\mathrm{d}x = \lim_{t\to 0} 2\sin\sqrt{x}\big|_t^1 = 2\sin(1) - \lim_{t\to 0} 2\sin\sqrt{t} = 2\sin(1).$$

(b) We integrate this by parts with $u = x \Rightarrow \mathrm{d}u = \mathrm{d}x$ and $\mathrm{d}v = e^{-x}\mathrm{d}x \Rightarrow v = -e^{-x}$,

$$\int_0^\infty xe^{-x}\mathrm{d}x = \lim_{t\to\infty}\int_0^t xe^{-x}\mathrm{d}x = \lim_{t\to\infty}\left[-xe^{-x}\big|_0^t + \int_0^t e^{-x}\mathrm{d}x\right]$$
$$= \lim_{t\to\infty}\left[-xe^{-x} - e^{-x}\right]_0^t = \lim_{t\to\infty} 1 - te^{-t} - e^{-t} = 1$$

We get this by employing,

$$\lim_{t\to\infty} te^{-t} = \lim_{t\to\infty}\frac{t}{e^t} = \lim_{t\to\infty}\frac{1}{e^t} = 0.$$

(6) We use disks to get $V = \pi \int_0^1 \frac{\mathrm{d}x}{(1+x^2)^2}$. Then we use trig-sub with $x = \tan\theta \Rightarrow \mathrm{d}x = \sec^2\theta \mathrm{d}\theta$,

$$\begin{aligned} V &= \pi \int_0^{\pi/4} \frac{\sec^2\theta \mathrm{d}\theta}{(1+\tan^2\theta)^2} = \pi \int_0^{\pi/4} \frac{\sec^2\theta}{\sec^4\theta}\mathrm{d}\theta = \pi \int_0^{\pi/4} \cos^2\theta \mathrm{d}\theta \\ &= \pi \int_0^{\pi/4} \frac{1}{2}(1+\cos 2\theta)\mathrm{d}\theta = \frac{\pi}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \frac{\pi}{2}\left[\frac{\pi}{4} + \frac{1}{2}\right] \end{aligned}$$

(7) Again, remember the standard forms of series.

(a) This is a typical limit comparison problem. Lets compare to $1/n$,

$$\lim_{n\to\infty} \frac{(n+1)/\sqrt{n^4+4}}{1/n} = \lim_{n\to\infty} \frac{n+1}{\sqrt{n^4+4}} \cdot n = \lim_{n\to\infty} \frac{n^2+n}{\sqrt{n^4+4}} = \lim_{n\to\infty} \frac{1+1/n}{\sqrt{1+4/n}} = 1.$$

So, this is a valid comparison. Since $\sum_{n=1}^{\infty} 1/n$ diverges by p-series because p = 1, $\sum_{n=1}^{\infty}(n+1)/\sqrt{n^4+4}$ diverges by the limit comparison test.

(b) We can use direct comparison for this. Notice $1/(e^n+1) \leq 1/e^n$, and since $\sum_{n=1}^{\infty} 1/e^n$ converges by geometric series because $1/e < 1$, $\sum_{n=1}^{\infty} 1/(e^n+1)$ converges by the direct comparison test.

(c) We can tell this diverges so lets just take the limit of the "n^{th}" term,

$$\lim_{n\to\infty} \frac{2^n+5^n}{4^n+5^n} = \lim_{n\to\infty} \frac{(2/5)^n+1}{(4/5)^n+1} = 1 \neq 0$$

And therefore it diverges.

(8) As per usual we apply ratio test,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^{n+1}(x-1)^{n+1}}{n+1} \cdot \frac{n}{3^n(x-1)^n}\right| = \lim_{n\to\infty} \frac{3n}{n+1}|x-1| = \lim_{n\to\infty} \frac{3|x-1|}{1+1/n} = 3|x-1|.$$

By the ratio test this needs to be less than 1 to converge absolutely, hence we require $|x-1| < 1/3$, i.e. the radius of convergence is $R = 1/3$. So the interval of absolute convergence is $2/3 < x < 4/3$. Now we test the end points. For $x = 4/3$ our series becomes $\sum_{n=1}^{\infty} 1/n$ which diverges by p-series because $p = 1$. For $x = 2/3$ we get $\sum_{n=1}^{\infty}(-1)^n/n$, which is an alternating series. We first take the limit of the "n^{th}" term, $\lim_{n\to\infty} 1/n = 0$. Next we show it's decreasing, $1/n > 1/(n+1)$. Therefore, the series converges by the alternating series test. This gives an interval of convergence of $2/3 \leq x < 4/3$.

(9) We know the Taylor series of $\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$.

(a) Plugging in x^2 and multiplying through by x gives,

$$x\cos x^2 = x\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+1}}{(2n)!}.$$

(b) We take one more term than the x^5 term and take the limit,

$$\lim_{x\to 0} \frac{x\cos(x^2 - x)}{3x^5} = \lim_{x\to 0} \frac{\left(x - \frac{x^5}{2} + \frac{x^9}{24} + \cdots\right) - x}{3x^5} = \lim_{x\to 0} \frac{-\frac{x^5}{2} + \frac{x^9}{24} + \cdots}{3x^5} = \frac{-1}{6}.$$

(10) (a) Notice $f^{(n)}(2) = e^2$, so we get,

$$e^x \approx e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 + \frac{e^2}{6}(x-2)^3.$$

(b) For the erorr we apply the Taylor remainder formula and we notice $\left|(x-2)^4\right| \leq 1$ in our interval.

$$|R_3| \leq \left|\frac{M}{4!}(x-2)^4\right| \leq \left|\frac{e^2}{4!}(x-2)^4\right| \leq \frac{e^2}{24}.$$