11.1 Parametrizations of Plane Curves

For parametric curves we think of a little creature on a 1-D curve or equivalently a bead on a wire. We denote these curves as $x = f(t)$ and $y = g(t)$ i.e. x and y as functions of t where we can think of t as the time the creature spends moving.

Ex: Consider $x = t^2 - 2t$, $y = t + 1$. We can convert this into x as a function of y, but now we have to be careful. When we do this we make sure that we adhere to the domain of t not y, so the domain of y must be restricted by the domain of t. We convert this by noting $t = y - 1$, and plugging in to the formula for x, $x = y^2 - 4y + 3$. We can sketch this, but we have to make sure we include the directions arrows. We can think of these arrows as the direction the creature is moving on the curve.

Many times we put restrictions on t, such as $a \leq t \leq b$ and $(f(a), f(b))$ is called initial point and $(f(b), g(b))$ is the terminal point.

(1) Consider $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$. We directly identify this as a circle. Notice that $x^2 + y^2 = 1$ and the direction of movement is counter-clockwise. What about $x = \cos 2t, \sin 2t$? In that case we go around twice.
(2) Consider $x = \sin t$, $y = \sin^2 t$. Notice that $y = x^2$ and $|x| \leq 1$, but this curve goes back and forth on this line because $\sin t$ is a periodic function, so the arrows must be in both directions.
(3) Consider $x = \cos 2t$, $y = \sin^2 t$. This gives $y = \frac{1}{2} - \frac{1}{2}\cos 2t = \frac{1}{2} - \frac{x}{2}$.

Know what Cardioids are.

Calculus with parametric curves and Applications

One thing we would like to do with parametric curves is find the slope at any point. It's quite easy to derive these:

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{g'(t)}{f'(t)};\ \frac{\mathrm{d}x}{\mathrm{d}t} \neq 0. \tag{1}$$

We can also calculate the second derivative:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}y'/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{g'(t)/f'(t)}{f'(t)} = \frac{g'(t)}{f'(t)^2};\ \frac{\mathrm{d}x}{\mathrm{d}t} \neq 0. \tag{2}$$

(1) A curve C is defined by $x = t^2$, $y = t^3 - 3t$.
  (a) Show the curve has two tangent lines at $(3, 0)$.
  (b) At what points are the tangent lines horizontal/vertical?
  (c) Where is the curve concave up/down?
  (d) Sketch (this was done in class).

(a) **Solution**: We notice that a 1-D curve can have two tangent lines only if it crosses itself in a transverse manner. This means there are two such times t where $(x, y) = (3, 0)$, so lets solve for this.

$$y = t^3 - 3t = t(t^2 - 3) = 0 \Rightarrow t = 0, \pm\sqrt{3},$$
$$x = t^2 = 3 \Rightarrow t = \pm\sqrt{3}.$$

Therefore, $(x, y) = (3, 0)$ when $t = \pm\sqrt{3}$ i.e. two different ts, and hence has two tangent lines.

(b) **Solution**: For the horizontal and vertical tangent lines lets equate the respective time derivative of x and y to zero.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2 - 3 = 0 \Rightarrow t = \pm 1,$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2t = 0 \Rightarrow t = 0.$$

Since there are no repeats we can say that the horizontal tangent lines occur at $t = \pm 1$ which means $(1, -2)$ and $(1, 2)$ and the vertical tangent line occurs at $t = 0$ which means $(0, 0)$.

(c) **Solution**: For the concavity we compute the second derivative, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{3(t^2+1)}{4t^3}$, so the curve is concave up for $t > 0$ and concave down for $t < 0$.

(2) Consider $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.

(a) Find the equation of the tangent line at $\theta = \pi/3$.

(b) At what points are the tangent lines horizontal/vertical?

(c) Find the area for the curve for $\theta \in [0, 2\pi]$.

(a) **Solution**: This is exactly like solving for the equation of a tangent line from Calc I, except we have a slightly different way of calculating the derivative. Lets go ahead and calculate the derivative,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{r\sin\theta}{r - r\cos\theta} = \frac{\sin\theta}{1 - \cos\theta}.$$

Then, at $y'(\pi/3) = \sqrt{3}$. Now we must find the respective x and y, which are $x(\pi/3) = r(\pi/3 - \sqrt{3}/2)$, $y(\pi/3) = r/2$. Now we plug this into the point slope form of the equation of a line,

$$y - y_0 = m(x - x_0) \Rightarrow y - \frac{r}{2} = \sqrt{3}\left(x - \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)r\right).$$

(b) **Solution**: To find the horizontal and vertical points we equate the relative derivatives to zero,

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = r\sin\theta = 0 \Rightarrow \theta = n\pi,$$
$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = r(1-\cos\theta) = 0 \Rightarrow \theta = 2n\pi.$$

Hence we get horizontal derivatives for every odd $n\pi$ i.e. $(2n-1)\pi$. Now for the even $n\pi$ both derivatives are zero, so we must take the limit,

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\theta=n\pi} = \lim_{\theta\to n\pi}\frac{\sin\theta}{1-\cos\theta} = \infty.$$

So, these are the vertical points. That means we get horizontal tangents at $((2n-1)\pi r, 2r)$ and vertical tangents at $(2n\pi r, 0)$.

(c) **Solution**: For the area lets not worry about the limits until we get to the parametric form. We start off with the usual integral and derive the parametric integral,

$$\int y\mathrm{d}x \int_0^{2\pi} y(\theta)\frac{\mathrm{d}x}{\mathrm{d}\theta}\mathrm{d}\theta = \int_0^{2\pi} r(1-\cos\theta)r(1-\cos\theta)\mathrm{d}\theta = r^2\int_0^{2\pi}(1-2\cos\theta+\cos^2\theta)\mathrm{d}\theta$$
$$= r^2\int_0^{2\pi}\left[1-2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right]\mathrm{d}\theta = r^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = 3\pi r^2.$$

If we recall, when we first learned arc length and surface area we derived our formulas through parametrization. At the time I said we didn't have to worry about it, but now this really comes into play. Since we already derived it I shall simply provide the formulas for arc length and surface area respectively,

$$\mathrm{L} = \int_\alpha^\beta \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\mathrm{d}t = \int_\alpha^\beta \sqrt{f'(t)^2+g'(t)^2}\mathrm{d}t. \tag{3}$$

$$\mathrm{SA} = \int_\alpha^\beta 2\pi y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\mathrm{d}t \qquad \text{Revolution about x-axis} \tag{4}$$

$$\mathrm{SA} = \int_\alpha^\beta 2\pi x\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\mathrm{d}t \qquad \text{Revolution about y-axis} \tag{5}$$

(1) Find the arc length of $x = \cos t$, $y = \sin t$ from $t = 0$ to $t = 2\pi$.
Solution: We first take the derivatives, $\mathrm{d}x/\mathrm{d}t = -\sin t$ and $\mathrm{d}y/\mathrm{d}t = \cos t$, then we plug into our formula,

$$\mathrm{L} = \int_0^{2\pi}\sqrt{\sin^2 t + \cos^2 t}\mathrm{d}t = \int_0^{2\pi}\mathrm{d}t = 2\pi.$$

(2) Find the arc length of $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$ from $t = 0$ to $t = 2\pi$.
Solution: Here we have the same procedure to get,

$$\begin{aligned} \text{L} &= \int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\mathrm{d}\theta = r\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\mathrm{d}\theta \\ &= r\int_0^{2\pi} \sqrt{2(2\sin^2\theta/2)}\mathrm{d}\theta = 2r\int_0^{2\pi} \sin\frac{\theta}{2}\mathrm{d}\theta = 2r\left[-2\cos\frac{\theta}{2}\right]_0^{2\pi} = 8r. \end{aligned}$$

(3) Show that the surface area of a sphere of radius r is $4\pi r^2$.
Solution: We can rotate the semicircle $x = r\cos t$, $y = r\sin t$ about the x-axis, hence

$$\text{SA} = \int_0^{\pi} 2\pi r\sin t\sqrt{(-r\sin t)^2 + (r\cos t)^2}\mathrm{d}t = 2\pi r^2\int_0^{\pi} \sin t\mathrm{d}t = -2\pi r^2\cos t|_0^{\pi} = 4\pi r^2.$$

11.3 and 11.4 Polar Coordinates and Sketching

For polar coordinates r is the distance from the origin and θ is the angle from the x-axis. The ordered pairs are denoted (r, θ) and $x = r\cos\theta$, $y = r\sin\theta$, which means $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$.

Ex: We plotted the following points in class:
a) $(1, 5\pi/4)$ b) $(2, 3\pi)$ c) $(2, -2\pi/3)$ d) $(-3, 3\pi/4)$.

Ex: Convert $(2, \pi/3)$ from Polar to Cartesian coordinates. Plugging these quantities into the formulas gives $(1, \sqrt{3})$.

Ex: Convert $(1, -1)$ from Cartesian to Polar coordinates. Plugging these quantities into the formulas gives $(\sqrt{2}, -\pi/4)$.

The main thing we will do with Polar coordinates is sketching and analyzing polar functions. A polar function is a function of the form $r = f(\theta)$ i.e. how the radius changes as a function of the angle.

We sketched $r = 2\cos\theta$, $r = 1 + \sin\theta$, and $r = \cos 2\theta$ in class. Refer to the link that I sent. I'll try and remember to send the link again with this email.

Ex: Consider $r = 1 + \sin\theta$, $0 \le \theta \le 2\pi$.
(a) Find the slope of the tangent line at $\theta = \pi/3$,
(b) Find the points where the tangent lines are horizontal/vertical.
(a) **Solution**: We must compute $\mathrm{d}y/\mathrm{d}x$ in the usual manner from last section,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta} = \frac{\cos\theta(1+2\sin\theta)}{1 - 2\sin^2\theta - \sin\theta} = \frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}.$$

Plugging in $\theta = \pi/3$ gives $\mathrm{d}y/\mathrm{d}x = -1$.

(b) **Solution**: We compute the respective derivatives and equate them to zero,

$$\frac{dy}{d\theta} = \cos\theta(1+2\sin\theta) = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}.$$
$$\frac{dx}{d\theta} = (1+\sin\theta)(1-2\sin\theta) = 0 \Rightarrow \theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}.$$

We can make conclusions about all the points except for the duplicate. For the duplicate we must take the limit

$$\lim_{\theta\to\frac{3\pi}{2}^-} \frac{dy}{dx} = -\lim_{\theta\to\frac{3\pi}{2}^+} \frac{dy}{dx} = \infty$$

So, the horizontal points correspond to $\theta = \pi/2$, $7\pi/6$, $11\pi/6$ and the vertical points correspond to $\theta = \pi/6$, $5\pi/6$, $3\pi/2$.

11.5 Areas and Lengths in Polar coordinates

We know the area of a sector is $A = r^2\theta/2$, so if $r = f(\theta)$ the area of a wedge in a polar curve will be,

$$\mathrm{A} = \int_\alpha^\beta \frac{1}{2} r^2 d\theta = \int_\alpha^\beta \frac{1}{2} f(\theta)^2 d\theta. \tag{6}$$

We can derive the arc length in the usual manner, but it does get very tedious, so one should go straight to the formula,

$$\mathrm{L} = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \tag{7}$$

(1) Find the area enclosed by $r = \cos 2\theta$ and the x-axis.
Solution: We plug this straight into the formula,

$$\mathrm{A} = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta d\theta = \int_0^{\pi/4} \cos^2 2\theta d\theta = \int_0^{\pi/4} \frac{1}{2}(1+\cos 4\theta) d\theta = \frac{1}{2}\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/4} = \frac{\pi}{8}.$$

(2) Find the area inside $r = 3\sin\theta$ and outside $r = 1+\sin\theta$.
Solution: The first thing we have to do is find where they intersect so that we can figure out our limits,

$$3\sin\theta = 1 + \sin\theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}.$$

Then, we plug into our formula,

$$\mathrm{A} = \frac{1}{2}\int_{\pi/6}^{5\pi/6} [(3\sin\theta)^2 - (1+\sin\theta)^2] d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (8\sin^2\theta - 1 - 2\sin\theta) d\theta$$
$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3 - 4\cos 2\theta - 2\sin\theta) d\theta = \frac{1}{2}[\theta - 2\sin 2\theta + 2\cos\theta]_{\pi/6}^{5\pi/6} = \pi.$$

(3) Find the length of $r = 1 + \sin\theta$ for $0 \leq \theta \leq 2\pi$.
Solution: We plug in to our formula to get,

$$\mathrm{L} = \int_0^{2\pi} \sqrt{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta} \mathrm{d}\theta = \int_0^{2\pi} \sqrt{2 + 2\sin\theta} \frac{\sqrt{2 - 2\sin\theta}}{2 - 2\sin\theta} \mathrm{d}\theta$$
$$= \int_0^{2\pi} \frac{\sqrt{4 - 4\sin^2\theta}}{\sqrt{2 - 2\sin\theta}} \mathrm{d}\theta = \int_0^{2\pi} \frac{2\cos\theta \mathrm{d}\theta}{\sqrt{2 - 2\sin\theta}}.$$

Now this is an improper integral at $\theta = \pi/2$, so we would need to take limits. Assuming we do this our final answer should be 8.