10.7 Power Series

A power series is the variable analog of a geometric series, and looks like this $\sum_{n=0}^{\infty} c_n x^n$. Notice that if we have the series $\sum_{n=0}^{\infty} x^n$, and we fix x, it is just a geometric series, so for $|x| < 1$, $\sum_{n=0}^{\infty} x^n = 1/(1-x)$. And we say that the series has a radius of convergence of $R = 1$.

We can also have a power series of the form, $\sum_{n=1}^{\infty} c_n(x-a)^n$, and this is called a power series centered at $x = a$. If $a = 0$, we are back to our usual power series, which we call a power series centered at $x = 0$, or simply a power series, i.e. if we don't mention what point it is centered around, it is centered at 0.

Ex: Our c_n need not be bounded. In this example we see what happens if we have unbounded c_n. Consider $\sum_{n=0}^{\infty} n!x^n$. We use the ratio test to see what our domain of convergence would be,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty}(n+1)|x| = \infty$$

So, the only way this will converge is when $x = 0$.

Ex: Now, lets find the domain of convergence of something more interesting. Consider the series $\sum_{n=1}^{\infty}(x-1)^n/n$. Lets apply ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1}\cdot\frac{n}{(x-3)^n}\right| = \frac{n}{n+1}|x-3| = \frac{1}{1+1/n}|x-3|.$$

Taking the limit gives, $\lim_{n\to\infty}\frac{1}{1+1/n}|x-3| = |x-3| < 3$, so the series converges absolutely for $2 < x < 4$. Now, we test the end points. Notice when $x = 2$ this is an alternating series and will converge by the alternating series test. However, for $x = 4$, it is a harmonic series and does not converge, by p-series since $p = 1$. If you have a question about the end points for this problem come ask me.

In the sequel we present a few important theorems that don't necessarily have to be memorized, but the ideas should be kept in mind.

Theorem 1. *Given $\sum_{n=0}^{\infty} c_n(x-a)^n$, we have three possibilities:*

(i) *The series converges at $x = a$,*
(ii) *The series converges for $|x-a| < R \in \mathbb{R}^+$,*
(iii) *The series converges for all x.*

Theorem 2. *If $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = c \neq 0$, then it converges absolutely for all $|x| < |c|$, and if it diverges at $x = d$, then it diverges for all $|x| > |d|$.*

Theorem 3. *Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$, $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converges absolutely for $|x| < R$, and $c_n = \sum_{n=0}^{\infty} a_k b_{n-k}$, then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$.*

Theorem 4. *Suppose $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$, then for all continuous functions f on $|f| < R$, $\sum_{n=0}^{\infty} a_n(f(x))^n$ also converges absolutely.*

Theorem 5. *Term by term differentiation: If $\sum c_n(x-a)^n$ has a radius of convergence, $R > 0$, it defines a function $f(x) = \sum c_n(x-a)^n$ on the interval $a - R < x < a + R$ and f has derivatives of all orders on $(a - R, a + R)$.*

Theorem 6. *Term by term integration: Suppose $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $a-R < x < a+R$; $R > 0$, then $\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$ converges in the same domain, and $\int f(x)dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$.*

(1) For this example we demonstrate the usefulness of term by term integration. Consider,

$$\frac{1}{x} = \frac{1}{1-(1-x)} = \sum_{n=0}^{\infty}(1-x)^n = 1 + (1-x) + (1-x)^2 + \cdots .$$

Integrating this term by term gives,

$$\ln x = \ln(1) + \sum_{n=0}^{\infty} \frac{(1-x)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(1-x)^{n+1}}{n+1} = (1-x) + \frac{1}{2}(1-x)^2 + \frac{1}{3}(1-x)^3 + \cdots .$$

And since the first series converges for $x \in (0, 2)$ so does the second series.

(2) Find the domain of convergence of $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.
Solution: We first apply ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n}.\right| = \left| -3x\sqrt{\frac{n+1}{n+2}}\right| = 3|x|\sqrt{\frac{1+1/n}{1+2/n}}.$$

Taking the limit gives, $\lim_{n\to\infty} |a_{n+1}/a_n| = 3|x| < 1$, hence $|x| < 1/3 = R$. Now, if $x = -1/3$ the series diverges by p-series because $p < 1$, and if $x = 1/3$ the series converges by the alternating series test. So, the domain of convergence is $-1/3 < x \leq 1/3$.

(3) Find the domain of convergence of $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$.
Solution: Again we apply ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n}\right| = \frac{1+1/n}{3}|x+2|.$$

Taking the limit gives, $\lim_{n\to\infty} |a_{n+1}/a_n| = \frac{1}{3}|x+2| < 1$, hence $|x+2| < 3$. Now, if $x = -5$ or $x = 1$, the series diverges by taking the limit of the a_n and showing it does not converge to 0.

10.8 Taylor Series

Suppose the function f has the following power series:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n.$$

Can we figure out what the coefficients are? Yes, yes we can. Notice that $f(a) = c_0$, so that gives us the first coefficient. For the second one lets differentiate, $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$. Now, if we plug in a we get $f'(a) = c_1$. How about the third? Well, $f''(x) = 2c_2 + 6c_3(x-a) + \cdots$, so $f''(a) = 2c_2$. Can we figure out what c_n should be? Well we see that if we keep taking derivatives and evaluating them at the center, we get $f^{(n)}(x) = n!c_n + \cdots$, so $c_n = f^{(n)}(x)/n!$. We have just derived a general formula for finding the coefficients of our series.

Theorem 7. *Suppose $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $|x-a| < R$. Then, $c_n = \frac{f^{(n)}(a)}{n!}$ and $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.*

Definition 1. The series representation

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2+\frac{f'''(a)}{3}(x-a)^3+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots \tag{1}$$

is called a Taylor series of f at $x = a$. If $a = 0$ we simply call this the Taylor series of f or the McLauren series of f - both are used interchangeably.

Ex: Find the Taylor series of $f(x) = e^x$ and it's radius of convergence.
Solution: This is easy because we can find the n^{th} derivative of e^x straight away, i.e. $f^{(n)}(x) = e^x$, hence $f^{(n)}(0) = 1$. So, $e^x = \sum_{n=0}^{\infty} x^n/n!$. Now, this is a power series so like any other power series we can find the radius of convergence by using either root or ratio test. Lets apply ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1}.$$

Taking the limit gives, $\lim_{n\to\infty} |a_{n+1}/a_n| = 0$, so $R = \infty$. Therefore, the Taylor series converges everywhere and it is an exact representation of e^x.

Definition 2. Let $\xi \in (a,b)$, and f have k derivatives on (a,b), then for any $n < k$ positive,

$$P_n(x) = f(\xi) + f'(\xi)(x-\xi) + \frac{f''(\xi)}{2}(x-\xi)^2 + \cdots + \frac{f^{(n)}(\xi)}{n!}(x-\xi)^n \tag{2}$$

is the n^{th} order Taylor polynomial of f at $x = \xi$.

Notice that the Taylor polynomial is just the truncated Taylor series. Since these are problems from the book, I'll just give you the solutions and you can look up the problems in the book.

7) We first evaluate the derivatives up to order $n = 3$: $f(\pi/4) = 1/\sqrt{2}$, $f'(\pi/4) = 1/\sqrt{2}$, $f''(\pi/4) = -1/\sqrt{2}$, and $f'''(\pi/4) = -1/\sqrt{2}$. Then, the Taylor polynomial is,

$$f(x) \approx P_3(x) = \frac{1}{\sqrt{2}}\left[1 + \left(x - \frac{\pi}{4}\right) - \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{6}\left(x - \frac{\pi}{4}\right)^3\right].$$

13) We just convert this into a power series,

$$f(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n.$$

19) Remind me to do this on Friday.

34) We want the first three terms, so lets find the first three terms of the respective series and then just subtract them,

$$f(x) = \cos x - \frac{2}{1-x} = \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) - \left(2 + 2x - 2x^2\right) = -1 - 2x - \frac{5x^2}{2}.$$