6.3 Arc Length

Arc length is just the sum of infinitesimally small pieces of an arc, so we can derive the formula:

$$L = \int_a^b \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}. \tag{1}$$

This can be parametrized in many ways, but there are two main ways:

$$L = \int_a^b \sqrt{1 + f'(x)^2}\,\mathrm{d}x \quad \text{if } f \in C^1([a,b]), \tag{2}$$

$$L = \int_c^d \sqrt{1 + g'(y)^2}\,\mathrm{d}y \quad \text{if } g \in C^1([c,d]). \tag{3}$$

This means that we use the first formula if $y = f(x)$ has a continuous derivative on $[a,b]$ (the interval between which we are calculating arc length), and we use the second formula if $x = g(y)$ has a continuous derivative on $[c,d]$. If it has a continuous derivative for both, we may use either formula.

(1) Find the arc length of $y^2 = x^3$ between $(1,1)$ and $(4,8)$.
Solution: We simply differentiate and plug into the formula. The derivative is $f'(x) = \frac{3}{2}x^{1/2}$, then

$$L = \int_1^4 \sqrt{1 + \frac{9}{4}x}\mathrm{d}x$$

We solve this via "u-sub", where $u = 1 + \frac{9}{4}x$, then

$$L = \frac{4}{9}\int_{13/4}^{10} \sqrt{u}\mathrm{d}u = \frac{1}{27}(80\sqrt{10} - 13\sqrt{13}).$$

(2) Find the arc length of $y^3 = x^2$ between $(0,0)$ and $(1,1)$.
Solution: We differentiate to get $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{3}x^{-1/3}$. This is clearly not continuous at $x = 0$, so we need to find another way. We can differentiate with respect to y instead, so $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{3}{2}y^{1/2}$, then

$$L = \int_0^1 \sqrt{1 + \frac{9}{4}y}\mathrm{d}y = \frac{4}{9}\int_1^{13/4} \sqrt{u} = \frac{1}{27}(13\sqrt{13} - 8).$$

(3) We can also think of arc length as a function. Suppose we want to figure out how the arc length changes as x varies for the curve $y^2 = x^3$ starting at $(0,0)$.
Solution: We can think of this problem as the arc length of a curve from $(0,0)$ to an arbitrary point, so we plug the derivative into the formula, but this time with respect to a dummy variable t,

$$s(x) = \int_0^x \sqrt{1 + \frac{9}{4}t}\mathrm{d}t = \int_1^{1+9x/4} \sqrt{u}\mathrm{d}u = \frac{2}{3}u^{3/2}\Big|_1^{1+9x/4} = \frac{2}{3}(1 + \frac{9}{4}x)^{3/2} - \frac{2}{3}.$$

We notice that we recover $s(1) = \frac{1}{27}(13\sqrt{13} - 8)$.

(4) Find the arc length function of $y = x^2 - \frac{1}{8}\ln x$ from $(1,1)$.
Solution: The derivative is:

$$f'(x) = 2x - \frac{1}{8x} \Rightarrow 1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 4x^2 + \frac{1}{2} + \frac{1}{64x^2} = (2t + \frac{1}{8t})^2.$$

Then we plug into the arc length formula,

$$s(x) = \int_1^x (2t + \frac{1}{8t})\mathrm{d}t = t^2 + \frac{1}{8}\ln t\Big|_1^x = x^2 + \frac{1}{8}\ln x - 1.$$

6.4 Areas of Revolution

Areas of revolution are similar to volumes of revolution, except now, we integrate over the length of the arc, so for a revolution about the x-axis we have

$$\mathrm{SA} = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\,\mathrm{d}x \tag{4}$$

And similarly for a revolution about the y-axis we have

$$\mathrm{SA} = 2\pi \int_c^d g(y)\sqrt{1 + g'(y)^2}\,\mathrm{d}y \tag{5}$$

However, just like for arc lengths, we can parametrize this in many different ways, but one convenient way to do it is as such:

$$\mathrm{SA} = 2\pi \int_c^d y\sqrt{1 + g'(y)^2}\,\mathrm{d}y$$
$$\mathrm{SA} = 2\pi \int_a^b x\sqrt{1 + f'(x)^2}\,\mathrm{d}x.$$

But, do not concern yourselves too much with these formulas, but it is important to realize there are many different ways we can solve the same problem.

(1) Find the surface area of $y = \sqrt{4 - x^2}$ bounded by $-1 \leq x \leq 1$ revolved about the x-axis.
Solution: We first differentiate to get $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x}{\sqrt{4-x^2}}$. Then we simply plug into the formula to get,

$$\mathrm{SA} = 2\pi \int_{-1}^1 \sqrt{4 - x^2}\sqrt{1 + \frac{x^2}{4 - x^2}}\mathrm{d}x = 2\pi \int_{-1}^1 \sqrt{4 - x^2 + (4 - x^2)\frac{x^2}{4 - x^2}}\mathrm{d}x = 2\pi \int_{-1}^1 2 = 8\pi.$$

(2) Find the surface area of the curve $y = x^2$ from $(1, 1)$ to $(2, 4)$ revolved about the y-axis.
Solution: Differentiating gives $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{2\sqrt{y}}$, then plugging into the formula gives

$$\mathrm{SA} = 2\pi \int_1^4 \sqrt{y}\sqrt{1 + \frac{1}{4y}}\mathrm{d}y = 2\pi \int_1^4 \sqrt{y + \frac{1}{4}}\mathrm{d}y.$$

This is solved via u-sub, where $u = y + \frac{1}{4}$, which gives

$$\mathrm{SA} = 2\pi \int_{5/4}^{17/4} \sqrt{u}\mathrm{d}u = \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5}).$$

6.5 Work

Work is the sum of forces exerted over a certain distance. It's induced by an action. For a constant force, it will be $W = Fd$, where W is the work, F is the force or equivalently the weight, and d is the distance traversed.

(1) How much work does it take to lift a 1.2-kg book .7m?
Solution: Assuming $g = 9.8$, the force is $F = mg = (1.2)(9.8)$, then the work is $W = Fd = (1.2)(9.8)(.7)$J.

(2) How much work does it take to lift a 6-lb weight 6ft?
Solution: Here we are already given the weight, which is equivalent to force, so $W = Fd = (6)(6) = 36$ft-lb.

Now, what if the force changes as a function? Then we need to add up all parts of the force, so we get the equation: $W = \int_a^b F(x)\mathrm{d}x$.

(3) Suppose the force is given by the function $f(x) = x^2 + 2x$-lb, where x is the distance the object is being moved. Find the work done to move this object from $x = 1$ft to $x = 3$ft.
Solution: We simply plug this into the formula for work to get $W = \int_1^3 (x^2 + 2x)\mathrm{d}x = \frac{1}{3}x^3 + x^2|_1^3 = \frac{50}{3}$.

(4) Consider a mass on a spring. It takes 40N to stretch it from it's natural length of 10cm to a length of 15cm. Find the work to stretch it from 15cm to 18cm.
Solution: We recall Hooke's law states that a mass on a spring feels a force proportional to the length the mass is stretched from the spring's natural position (i.e. $F = kx$), where x is the distance from the natural position and k is the spring constant.

Notice the natural length is at 10cm and the length that it is stretched to is 15cm, so lets denote the natural length as x_* and the initial stretched length as x_0, then $x_* = 0$ because the natural length is always defined to be at $x = 0$ and $x_0 = 5$cm because it is 5cm away from the natural length.

Also notice, the units are in cm, so we must change them to the proper SI units of meters: $x_0 = 5\text{cm} = .05\text{m}$ (the initial length) and $x_1 = 8\text{cm} = .08\text{m}$ (the length we want to stretch it to). First, we need to back out the spring constant,

$$f(x_0) = kx_0 = k(.05) = 40 \Rightarrow k = 800 \Rightarrow F(x) = 800x.$$

Now, we just plug this into the formula for work to get,

$$W = \int_{.05}^{.08} 800x\mathrm{d}x = 400x^2|_{.05}^{.08} = 1.56\text{J}.$$

Tougher examples.

We may not always have a problem where we can put quantities into a set formula. Many times we'll have to derive our own formula. These are the sort of problems they love to put on exams.

(1) Consider a 200lb - 100ft long cable hanging off the top of a building. Find the work it takes to pull the entire cable to the top of the building.
Solution: Lets define our coordinate system to be 0 at the top and 100 at the bottom. Now, lets break the cable up into i pieces, such that the i^{th} piece is at a distance x_i from the top and has a length of Δx_i. Notice the cable is 2lbs per ft, so the i^{th} cable weighs $F = 2\Delta x_i$. Then, the work it takes to move the i^{th} piece to the top is $W_i \approx 2x_i\Delta x_i$. Now we sum these pieces up and take the limit as $\Delta x_i \to 0$ in order to get the exact work done. This gives us an integral,

$$W = \int_0^{100} 2x\mathrm{d}x = x^2\Big|_0^{100} = 10^4 \text{ ft-lb}.$$

(2) Consider a conical tank where the base is on top and the point is on the bottom. It has a radius of $r = 4$m and a height of 10m. Suppose the tank is filled to 8m from the bottom with water. Assuming water has a density of $\rho = 1000$ kg/m^3, find the work it takes to pump all the water out of the tank from the top.
Solution: Lets define our coordinate system to be 0 at the top and 10 at the bottom, so our water level is at $x = 2$. We break the tank up into circular cylinders, such that the i^{th} cylinder has a height of Δx_i and a radius of r_i. We need to find r_i in terms of x_i. We can do this by using similar triangles, i.e. the ratio of the radii will be equivalent to the ratio of the heights of the big triangle (half the cross-section of the tank) and the small triangle (half the cross-section of the water). The ratio is:

$$\frac{r_i}{4} = \frac{10 - x_i}{10} \Rightarrow r_i = \frac{2}{5}(10 - x_i).$$

Now we know the volume of the i^{th} cylinder will be,

$$V_i \approx \pi r_i^2 \Delta x_i = \frac{4\pi}{25}(10 - x_i)^2\Delta x.$$

Then, this has a mass of

$$M_i \approx \rho V_i \approx 1000\frac{4\pi}{25}(10 - x_i)^2\Delta x_i = 160\pi(10 - x_i)^2\Delta x_i.$$

From this we get the weight,

$F_i \approx M_i g \approx (9.8)160\pi(10 - x_i)^2 \Delta x_i \approx 1570\pi(10 - x_i)^2 \Delta x_i.$

From this we get the work done to move the i^{th} cylinder to the top,

$$W_i \approx F_i x_i \approx 1570\pi x_i(10 - x_i)^2 \Delta x_i.$$

Taking the limit gives us the near-exact value of,

$$W \approx \int_2^1 01570\pi x_i(10 - x_i)^2 \Delta x_i \approx 3.4 \text{ x } 10^6 \text{ J}$$