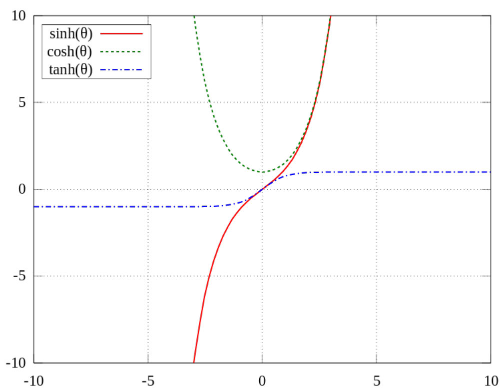
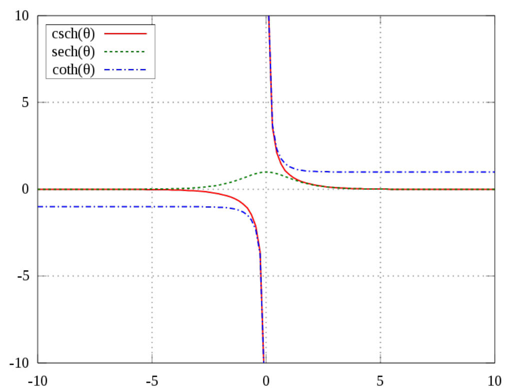
7.3 HYPERBOLIC FUNCTIONS

Hyperbolic functions are similar to trigonometric functions, and have the following definitions:

- $\sinh x = \frac{1}{2}(e^x - e^{-x})$
- $\cosh x = \frac{1}{2}(e^x + e^{-x})$
- $\tanh x = \frac{\sinh x}{\cosh x}$
- $\text{csch} x = 1/\sinh x$
- $\text{sech} x = 1/\cosh x$
- $\coth x = 1/\tanh x$

It's also useful to know what they look like





To remember what they look like, just use the definitions and recall what the exponential functions look like and take the average. If you're confused as to what I'm talking about make sure to ask me to explain it.

They are subject to the following identities:

- $\sinh(-x) = -\sinh x$
- $\cosh(-x) = \cosh x$
- $\cosh^2 x - \sinh^2 x = 1$
- $1 - \tanh^2 x = \text{sech}^2 x$
- $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
- $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

We can prove some of these things, so we may get a better understanding of the identities. Proofs are important, even for engineers!

Theorem 1. $\cosh^2 x - \sinh^2 x = 1$.

Proof. We go straight to the definition,

$$\cosh^2 x - \sinh^2 x = \left[\frac{1}{2}(e^x + e^{-x})\right]^2 - \left[\frac{1}{2}(e^x - e^{-x})\right]^2$$
$$= \frac{1}{4}(e^{2x} + 2 + e^{-2x}) - \frac{1}{4}(e^{2x} - 2 + e^{-2x}) = 1$$

□

Theorem 2. $1 - \tanh^2 x = \text{sech}^2 x$

Proof. Here we simply divide the entire equation by $\cosh^2 x$,

$$\left[\cosh^2 x - \sinh^2 x = 1\right] \frac{1}{\cosh^2 x} \Rightarrow 1 - \tanh^2 x = \text{sech}^2 x.$$

The other identities are proved similar to this one. If you have time, you should try to prove the other identities by yourselves. Even though they wont appear on exams they will help you get a better understanding of the concepts.

□

Here is a nice proof of one of the most important trigonometric identities, and all other identities can be very easily derived through these identities in a similar fashion to the above theorem.

Theorem 3. $\sin^2\theta + \cos^2\theta = 1$.

Proof. Consider a right triangle and one non-right angle θ. Let the side opposite to θ be of length x, the side adjacent to θ be of length y, and the hypotenuse z. Then, $\sin\theta = x/z$ and $\cos\theta = y/z$, and by the Pythagorean theorem $x^2 + y^2 = z^2$, then

$$\sin^2\theta + \cos^2\theta = \frac{x^2}{z^2} + \frac{y^2}{z^2} = \frac{x^2 + y^2}{z^2} = \frac{z^2}{z^2} = 1.$$

□

It is important to know the derivatives of hyperbolic functions as well,

- $(\sinh x)' = \cosh x$
- $(\cosh x)' = \sinh x$
- $(\tanh x)' = \text{sech}^2 x$
- $(\text{csch} x)' = -\text{csch} x \coth x$
- $(\text{sech} x)' = -\text{sech} x \tanh x$
- $(\coth x)' = -\text{csch}^2 x$

These can all be derived very easily from the definitions.

Ex: $(\cosh \sqrt{x})' = \frac{1}{2\sqrt{x}}(\sinh(\sqrt{x}))$.

We also have inverse hyperbolic functions, which are far less straightforward to derive. However, the derivations of these functions are quite cute, so you should try them. I will derive $\sinh^{-1} x$ here, and then just provide formulas for the rest.

Theorem 4. $\sinh^{-1} = \ln(x + \sqrt{x^2+1})$.

Proof. We begin with the definition of $y = \sinh x$. Recall in order to get the inverse we solve for x.

$$\frac{1}{2}(e^x - e^{-x}) = y \Rightarrow e^x - e^{-x} = 2y \Rightarrow e^x - 2y - e^{-x} = 0$$

If we multiply the equation through by e^x we get,

$$e^{2x} - 2ye^x - 1 = 0.$$

Notice, this is precisely the form of the quadratic polynomial: $ax^2 + bx + c = 0$, so we can solve this via the quadratic formula,

$$e^x = \frac{1}{2}(2y \pm \sqrt{4y^2+4}) = y \pm \sqrt{y^2+1} \Rightarrow x = \ln(y \pm \sqrt{y^2+1}).$$

Now, we choose the "+" branch because if we chose "-" the argument of the log would be negative. □

The other ones are derived in a similar fashion, and you are encouraged to try them out. While these sort of derivations will never appear on an exam, understanding the derivations will help you become a better engineer, programmer,etc.

- $\sinh^{-1} = \ln(x + \sqrt{x^2+1})$
- $\cosh^{-1} = \ln(x + \sqrt{x^2-1}); x \geq 1$
- $\tanh^{-1} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right); |x| < 1$
- $\coth^{-1} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right); |x| > 1$
- $\text{sech}^{-1} = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right); 0 < x \leq 1$
- $\text{csch}^{-1} = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right); x \neq 0$

It is useful to know the derivatives of the inverses in order to solve various integrals. Again, they aren't the most straightforward to derive, but lets derive it for $\sinh^{-1} x$,

Theorem 5. $\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$

Proof. As usual, we start from the definition,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1} x &= \frac{\mathrm{d}}{\mathrm{d}x}\ln(x+\sqrt{x^2+1}) = \frac{1+\frac{1}{2}(x^2+1)^{-1/2}(2x)}{x+(x^2+1)^{1/2}} \\ &= \frac{1+x(x^2+1)^{-1/2}}{x+(x^2+1)^{1/2}}\cdot\frac{(x^2+1)^{1/2}}{(x^2+1)^{1/2}} \\ &= \frac{(x^2+1)^{1/2}+x}{x+(x^2+1)^{1/2}}\cdot\frac{1}{(x^2+1)^{1/2}} = \frac{1}{\sqrt{x^2+1}}.\end{aligned}$$

□

The rest are,

- $\frac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$
- $\frac{\mathrm{d}}{\mathrm{d}x}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}; x > 0$
- $\frac{\mathrm{d}}{\mathrm{d}x}\tanh^{-1} x = \frac{1}{1-x^2}; |x| < 1$
- $\frac{\mathrm{d}}{\mathrm{d}x}\coth^{-1} x = \frac{1}{1-x^2}; |x| > 1$
- $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{sech}^{-1} x = \frac{1}{x\sqrt{1-x^2}}; 0 < x < 1$
- $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{csch}^{-1} x = \frac{1}{|x|\sqrt{1+x^2}}; x \neq 0$

8.1 Integration by parts

The modern notion of integration by parts comes from a beautiful theory of integrals by Riemann and Stieltjes in 1894, soon after which Stieltjes passed away. The idea is we can integrate over certain functions instead of just over x. We can think of it as a generalization of "u-sub".

To derive it, consider the product rule,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x) \Rightarrow \mathrm{d}[f(x)g(x)] = f(x)g'(x)\mathrm{d}x + g(x)f'(x)\mathrm{d}x \\
&\Rightarrow \int \mathrm{d}[f(x)g(x)] = f(x)g(x) = \int f(x)g'(x)\mathrm{d}x + \int g(x)f'(x)\mathrm{d}x \\
&\Rightarrow \int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int g(x)f'(x)\mathrm{d}x.
\end{aligned}$$

This can be written in the form, which we will use from now on

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u. \tag{1}$$

(1) $I = \int x \sin x \mathrm{d}x$.
Solution: Let $u = x \Rightarrow \mathrm{d}u = \mathrm{d}x$ and $\mathrm{d}v = \sin x \Rightarrow v = -\cos x$. Then,

$$I = -x\cos x + \int \cos x \mathrm{d}x = -x\cos x + \sin x + C.$$

We see here that we generally choose the easiest thing to integrate as $\mathrm{d}v$. We can use ILATE: InverseLogsAlgebraicTrigonometricExponential, to help determine which is easier to integrate. Things get easier to integrate as we go to the right, for example, Exponentials are easier to integrate than Trigonometric functions. But this doesn't always work! So, only use it as a guide, not a rule of thumb.

(2) $I = \int \ln x \mathrm{d}x$.
Solution: Let $u = \ln x \Rightarrow \mathrm{d}u = \frac{1}{x}\mathrm{d}x$ and $\mathrm{d}v = \mathrm{d}x \Rightarrow v = x$. Then,

$$I = x\ln x - \int x\frac{\mathrm{d}x}{x} = x\ln x - x + C.$$

(3) $I = \int t^2 e^t \mathrm{d}t$.
Solution: Let $u = t^2 \Rightarrow \mathrm{d}u = 2t\mathrm{d}t$ and $\mathrm{d}v = e^t\mathrm{d}t \Rightarrow v = e^t$. Then,

$$I = t^2 e^t - 2\int te^t \mathrm{d}t.$$

Notice, we need to integrate by parts again for the second integral $I_2 = \int te^t \mathrm{d}t$. Let $u = t \Rightarrow \mathrm{d}u = \mathrm{d}t$ and $\mathrm{d}v = e^t\mathrm{d}t \Rightarrow v = e^t$. Then

$$I_2 = te^t - \int e^t \mathrm{d}t = te^t - e^t.$$

Plugging this back into I gives,

$$I = t^2 e^t - 2te^t + 2e^t + C.$$

It may be appealing to do this sort of problem using "tabular integration", however you should avoid using this "method". If you make a mistake using this "method", you will lose a majority of the points. You are better off doing integration by parts twice.

(4) $I = e^x \sin x \mathrm{d}x$.
Solution: Let $u = e^x\mathrm{d}x \Rightarrow \mathrm{d}u = e^t\mathrm{d}t$ and $\mathrm{d}v = \sin x \Rightarrow v = -\cos x$. Then,

$$I = -e^x \cos x + \int e^x \cos x \mathrm{d}x.$$

We must do another integration by parts on the second integral. Let $u = e^x \Rightarrow \mathrm{d}u = e^x\mathrm{d}x$ and $\mathrm{d}v = \cos x \Rightarrow v = \sin x$. Then,

$$I_2 = e^x \sin x - \int e^x \sin x \mathrm{d}x.$$

Plugging this into I gives,

$$I = e^x \sin x - e^x \cos x - \int e^x \sin x \mathrm{d}x.$$

Now, we add both sides by $\int e^x \sin x \mathrm{d}x$, to get

$$2\int e^x \sin x \mathrm{d}x = e^x \sin x - e^x \cos x \Rightarrow \int e^x \sin x \mathrm{d}x = \frac{1}{2}(e^x \sin x - e^x \cos x) + C.$$

Notice, for this problem it didn't matter if you chose e^x or $\sin x$ and $\cos x$ as your u or $\mathrm{d}v$. Try this problem the other way around to convince yourself that it works both ways. And as usual, if you're confused about what I'm talking about, please make sure to ask me. It's better to get questions answered early on before you're bombarded with new material.

(5) $I = \int_0^1 \tan^{-1} x \mathrm{d}x$.
Solution: Let $u = \tan^{-1} x \Rightarrow \frac{\mathrm{d}x}{1+x^2}$ and $\mathrm{d}v = \mathrm{d}x \Rightarrow v = x$. Then,

$$I = x \tan^{-1} x\Big|_0^1 - \int_0^1 \frac{x\mathrm{d}x}{1+x^2}.$$

The second integral is our usual u-sub integral where $u = 1 + x^2 \Rightarrow \mathrm{d}u = 2x\mathrm{d}x$. Then,

$$I_2 = \frac{1}{2}\int_1^2 \frac{\mathrm{d}u}{u} = \frac{1}{2}\ln u\Big|_1^2 = \ln 2.$$

Plugging this back into I gives,

$$I = x\tan^{-1} x\Big|_0^1 - \frac{1}{2}\ln u\Big|_1^2 = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

(6) This next example is a test of our abilities to think abstractly. You wont see this sort of thing on the exam, but you'll see things on the exam that use many of the tricks we will use on this example.
Find a reduction formula for $I = \int \sin^n x \mathrm{d}x$.
Solution: Let $u = \sin^{n-1} x \Rightarrow \mathrm{d}u = (n-1)\sin^{n-2} x \cos x \mathrm{d}x$ and $\mathrm{d}v = \sin x \mathrm{d}x \Rightarrow v = -\cos x$. Then,

$$\begin{aligned}
\int \sin^n x\mathrm{d}x &= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \cos^2 x\mathrm{d}x \\
&= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\mathrm{d}x \\
&= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x - (n-1)\int \sin^n x\mathrm{d}x \\
&\Rightarrow n\sin^n x\mathrm{d}x = -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} x\mathrm{d}x \\
&\Rightarrow \sin^n x\mathrm{d}x = -\frac{1}{n}\cos x\sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\mathrm{d}x.
\end{aligned}$$