8.2 Trigonometric Integration

Lets look at a few examples first and then we'll develop a general strategy.

Sines and Cosines.

(1) Consider $\int \cos^3 x \mathrm{d}x$.
Solution: We recall the identity: $\cos^2 = 1 - \sin^2 x$. Lets see if we can use this to simplify the problem.

$$\int \cos^3 x \mathrm{d}x = \int \cos x[1 - \sin^2 x]\mathrm{d}x = \int \cos x \mathrm{d}x - \int \sin^2 x \cos x \mathrm{d}x.$$

Now, the first integral is easy and the second integral we solve via u-sub where $u = \sin x \Rightarrow \mathrm{d}u = \cos x \mathrm{d}x$.

$$\int \cos^3 x \mathrm{d}x = \sin x - \int u^2 \mathrm{d}u = \sin x - \frac{1}{3}u^3 + C = \sin x - \frac{1}{3}\sin^3 x + C$$

(2) $\int \sin^5 x \cos^2 x \mathrm{d}x$.
Solution: Lets use the same strategy as above, except this time on $\sin x$.

$$\int \sin^5 x \cos^2 x \mathrm{d}x = \int (\sin^2 x)^2 \sin x \cos^2 x \mathrm{d}x = \int (1 - \cos^2 x)^2 \cos^2 x \sin x \mathrm{d}x.$$

We can go straight to u-sub with $u = \cos x \Rightarrow \mathrm{d}u = -\sin x$,

$$\int \sin^5 x \cos^2 x \mathrm{d}x = -\int (1-u^2)^2 u^2 \mathrm{d}u = -\frac{1}{3}u^3 + \frac{2}{5}u^5 - \frac{1}{7}u^7 + C = -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$$

(3) $\int_0^\pi \sin^2 x \mathrm{d}x$
Solution: For this problem if we used the identity we used for the past two problems we would be going in circles, so we use another identity - the double angle formula: $\cos 2x = 1 - 2\sin^2 x$,

$$\int_0^\pi \sin^2 x \mathrm{d}x = \frac{1}{2}\int_0^\pi (1 - \cos 2x)\mathrm{d}x = \left[\frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)\right]_0^\pi = \frac{\pi}{2}$$

(4) $\int \sin^4 x \mathrm{d}x$
Solution: This is similar to the above problem,

$$\int \sin^4 x \mathrm{d}x = \int \left[\frac{1}{2}(1 - \cos 2x)\right]^2 \mathrm{d}x = \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\mathrm{d}x$$
$$= \frac{1}{4}\int \left[1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right]\mathrm{d}x = \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right) + C$$

Strategies for $\int \sin^m x \cos^n x\,dx$.

(1) If the power of the cosine term is odd (i.e. $n = 2k+1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$,

$$\begin{aligned}\int \sin^m x \cos^{2k+1} \mathrm{d}x &= \int \sin^m x (\cos^2 x)^k \cos x \mathrm{d}x \\ &= \int \sin^m x (1 - \sin^2 x)^k \cos x \mathrm{d}x. \qquad (1)\end{aligned}$$

Then substitute $u = \sin x \Rightarrow \mathrm{d}u = \cos x$.

(2) If the power of the sine term is odd (i.e. $m = 2k+1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$,

$$\int \sin^{2k+1} \cos^n x \mathrm{d}x = \int (\sin^2 x)^k \cos^n x \mathrm{d}x = \int (1 - \cos^2 x)^k \cos^n x \sin x \mathrm{d}x. \qquad (2)$$

Then substitute $u = \cos x \Rightarrow \mathrm{d}u = -\sin x$.

(3) If the powers of both sine and cosine are even, use the double-angle formulas:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x) \qquad \sin x \cos x = \frac{1}{2}\sin 2x.$$

Tangents and Secants.

(1) $\int \tan^6 x \sec^4 x \mathrm{d}x$.
Solution: We recall the identity $\sec^2 x = 1 + \tan^2 x$, and see where this takes us

$$\int \tan^6 x \sec^4 x \mathrm{d}x = \int \tan^6 x (1 + \tan^2 x) \sec^2 x \mathrm{d}x.$$

Then we substitute $u = \tan x \Rightarrow \mathrm{d}u = \sec^2 x \mathrm{d}x$, then

$$\int \tan^6 x \sec^4 x \mathrm{d}x = \int u^6 (1 + u^2) \mathrm{d}u = \frac{1}{7}u^7 + \frac{1}{9}u^9 + C = \frac{1}{7}\tan^7 x + \frac{1}{9}\tan^9 x + C$$

(2) $\int \tan^5 \theta \sec^7 \theta \mathrm{d}\theta$.
Solution: Here lets try using the other identity: $\tan^2 x = \sec^2 x - 1$,

$$\int \tan^5 \theta \sec^7 \theta \mathrm{d}\theta = \int \tan^4 \theta \sec^6 \theta \sec \theta \tan \theta \mathrm{d}\theta = \int (\sec^2 \theta - 1)^2 \sec^6 \theta \sec \theta \tan \theta \mathrm{d}\theta.$$

We employ the u-sub $u = \sec \theta \Rightarrow \mathrm{d}u = \sec \theta \tan \theta \mathrm{d}\theta$,

$$\int \tan^5 \theta \sec^7 \theta \mathrm{d}\theta = \int (u^2 - 1)^2 u^6 \mathrm{d}u = \frac{1}{11}u^{11} - \frac{2}{9}u^9 + \frac{1}{7}u^7 + C = \frac{1}{11}\sec^{11} x - \frac{2}{9}\sec^9 x + \frac{1}{7}\sec^7 x + C$$

Strategies for $\int \tan^m x \sec^n x dx$.

(1) If the power of the secant term is even (i.e. $n = 2k$, $k \geq 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$,

$$\begin{aligned}\int \tan^m x \sec^{2k} x \mathrm{d}x &= \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x \mathrm{d}x \\ &= \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x \mathrm{d}x. \qquad (3)\end{aligned}$$

Then substitute $u = \tan x \Rightarrow \mathrm{d}u = \sec^2 x \mathrm{d}x$.

(2) If the power of the tangent term is odd (i.e. $m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$,

$$\begin{aligned}\int \tan^{2k+1} x \sec^n x \mathrm{d}x &= \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x \mathrm{d}x \\ &= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \mathrm{d}x. \qquad (4)\end{aligned}$$

Then substitute $u = \sec x \Rightarrow \mathrm{d}u = \sec x \tan x \mathrm{d}x$

Useful Integrals.
These integrals are also pretty easy to derive if you forget them,

$$\int \tan x \mathrm{d}x = -\ln|\cos x| + C = \ln|\sec x| + C. \qquad (5)$$

$$\int \sec x \mathrm{d}x = \ln|\sec x + \tan x| + C. \qquad (6)$$

(1) $\int \tan^3 x \mathrm{d}x$.
Solution: We use the identity $\tan^2 x = \sec^2 x - 1$,

$$\int \tan^3 x \mathrm{d}x = \int \tan x (\sec^2 x - 1) \mathrm{d}x = \frac{1}{2} \tan^2 x - \ln|\sec x| + C.$$

(2) $\int \sec^3 x \mathrm{d}x$.
Solution: We integrate by parts with $u = \sec x \Rightarrow \mathrm{d}u = \sec x \tan x \mathrm{d}x$ and $\mathrm{d}v = \sec^2 x \Rightarrow v = \tan x$, then

$$\begin{aligned}\int \sec^3 x \mathrm{d}x &= \sec x \tan x - \int \sec x \tan^2 x \mathrm{d}x = \sec x \tan x - \int \sec x (\sec^2 x - 1) \mathrm{d}x \\ &= \sec x \tan x - \int \sec^3 x \mathrm{d}x + \int \sec x \mathrm{d}x = \sec x \tan x - \int \sec^3 x \mathrm{d}x + \ln|\sec x + \tan x| \\ &\Rightarrow \int \sec^3 x \mathrm{d}x = \frac{1}{2}[\sec x \tan x + \ln|\sec x + \tan x|] + C.\end{aligned}$$

Useful identities that you probably wont have to use that much.

$$\sin a \cos b = \frac{1}{2}[\sin(a-b) + \sin(a+b)] \tag{7}$$

$$\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)] \tag{8}$$

$$\cos a \cos b = \frac{1}{2}[\cos(a-b) + \cos(a+b)]. \tag{9}$$

(1) $\int \sin 4x \cos 5x \mathrm{d}x$.

Solution: We use the first identity to get,

$$\int \sin 4x \cos 5x \mathrm{d}x = \frac{1}{2}\int (\sin(-x) + \sin 9x)\mathrm{d}x = \frac{1}{2}\cos x - \frac{1}{18}\cos 9x + C.$$

One can also do this problem by parts, which is actually the preferred method, but a little extra knowledge never hurt anyone.