8.3 TRIGONOMETRIC SUBSTITUTIONS

Lets begin with an example,

Ex: Consider $\int \sqrt{1-x^2}\mathrm{d}x$.
Solution: This reminds us of the identity $1-\sin^2\theta = \cos^2\theta$, so lets use the substitution $x = \sin\theta \Rightarrow \mathrm{d}x = \cos\theta \mathrm{d}\theta$.

$$\int \sqrt{1-x^2}\mathrm{d}x = \int \sqrt{1-\sin^2\theta}\cos\theta\mathrm{d}\theta = \int \sqrt{\cos^2\theta}\cos\theta\mathrm{d}\theta = \int \cos^2\theta\mathrm{d}\theta$$
$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C = \frac{\theta}{2} + \sin\theta\cos\theta = \frac{1}{2}\sin^{-1}x + x\sqrt{1-x^2}.$$

The tricky part is going from θ to x. We see that since $x = \sin\theta$, $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$. We can also use our right triangles to help us.

Let us employ a table of trig substitutions that will help us with the decision making process.

Expression	Substitution	Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta$, $-\pi/2 \le \theta \le \pi/2$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$, $-\pi/2 < \theta < \pi/2$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$, $0 \le \theta < \pi/2$, $\pi \le \theta < 3\pi/2$	$\sec^2\theta - 1 = \tan^2\theta$

(1) $I = \int \frac{\sqrt{9-x^2}}{x^2}\mathrm{d}x$.
Solution: This is of the form of the first case, so we use the substitution: $x = 3\sin\theta \Rightarrow \mathrm{d}x = 3\cos\theta\mathrm{d}\theta$,

$$I = \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}3\cos\theta\mathrm{d}\theta = \int \frac{9\cos^2\theta}{9\sin^2\theta}\mathrm{d}\theta = \int \cot^2\theta\mathrm{d}\theta = \int(\csc^2\theta - 1)\mathrm{d}\theta = -\cot\theta - \theta + C.$$

Now we must plug back in for θ. Since $x = 3\sin\theta$, $\sin\theta = x/3$, and we recall that sine is opposite over hypotenuse and cotangent is adjacent over opposite. We may denote the opposite side as x and the hypotenuse side as 3, then by the Pythagorean theorem the adjacent side is $\sqrt{9-x^2}$. This gives us, $\cot\theta = \sqrt{9-x^2}/x$, then

$$I = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

Note that we have to substitute back in only for indefinite integrals. For definite integrals it's easier to just change the limits.

(2) $I = \int \frac{\mathrm{d}x}{x^2\sqrt{x^2+4}}$.
Solution: This is of the form of the second case, so we substitute $x = 2\tan\theta \Rightarrow \mathrm{d}x = 2\sec^2\theta \mathrm{d}\theta$,

$$I = \int \frac{2\sec^2\theta \mathrm{d}\theta}{4\tan^2\theta\sqrt{4\tan^2\theta+4}} = \frac{1}{4}\int \frac{\sec\theta \mathrm{d}\theta}{\tan^2\theta} = \frac{1}{4}\int \frac{\cos\theta \mathrm{d}\theta}{\sin^2\theta}.$$

We solve this via u-sub with $u = \sin\theta \Rightarrow \mathrm{d}u = \cos\theta \mathrm{d}\theta$.

$$I = \frac{1}{4}\int \frac{\mathrm{d}u}{u^2} = -\frac{1}{4u} + C = -\frac{1}{\sin\theta} + C.$$

Since $x = 2\tan\theta$, $\tan\theta = x/2$, so $\sin\theta = x/\sqrt{x^2+4}$, then

$$I = -\frac{\sqrt{x^2+4}}{4x} + C.$$

(3) $I = \int \frac{x\mathrm{d}x}{\sqrt{x^2+4}}$.
Solution: Thought we had to use a trig substitution didn't ya? NOPE! U-Sub! Let $u = x^2 + 4 \Rightarrow \mathrm{d}u = 2x\mathrm{d}x$.

$$I = \frac{1}{2}\int \frac{\mathrm{d}u}{\sqrt{u}} = \sqrt{u} + C = \sqrt{x^2+4} + C.$$

This shows us that if we take a few seconds to think about a problem we can find a much easier solution.
(4) $I = \int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}$.
Solution: This is of the form of the third case, so we substitute $x = a\sec\theta \Rightarrow \mathrm{d}x = a\sec\theta\tan\theta$.

$$I = \int \frac{a\sec\theta\tan\theta \mathrm{d}\theta}{\sqrt{a^2\sec^2\theta - a^2}} = \int \frac{a\sec\theta\tan\theta \mathrm{d}\theta}{a\tan\theta} = \int \sec\theta \mathrm{d}\theta = \ln|\sec\theta + \tan\theta| + C.$$

Since $x = a\sec\theta$, $\sec\theta = x/a$, so $\tan\theta = \sqrt{x^2-a^2}/a$, then

$$I = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C.$$

Notice, $a\cosh x$ is an equivalent answer, but the preferable method is the way it was done here.

(5) $\int_0^{3\sqrt{3}/2} \frac{x^3 \mathrm{d}x}{(4x^2+9)^{3/2}}$.

Solution: This is of the form of the second case, but here we have a coefficient in front of the x term. We can either pull the 4 out and then start our calculations or we can see what x has to be with the 4 there. It's much easier to come up with a substitution for x that produces a desired result than to pull the coefficient out. Notice that we need the coefficient in front of the $\tan^2$ term (after substitution of course) to be 9, so we have that $4x^2 = 9\tan^2\theta$, then $x = (3/2)\tan\theta \Rightarrow \mathrm{d}x = (3/2)\sec^2\theta \mathrm{d}\theta$.

$$I = \left(\frac{3}{2}\right)^4 \int_0^{\pi/3} \frac{\tan^3\theta \sec^2\theta \mathrm{d}\theta}{(9\tan^2\theta+9)^{3/2}} = \left(\frac{3}{2}\right)^4 \int_0^{\pi/3} \frac{\tan^3\theta\sec^2\theta\mathrm{d}\theta}{3^3\sec^3\theta} = \frac{3}{16}\int_0^{\pi/3}\frac{\tan^3\theta}{\sec\theta}\mathrm{d}\theta$$

$$= \frac{3}{16}\int_0^{\pi/3}\frac{\sin^3\theta}{\cos^2\theta}\mathrm{d}\theta = \frac{3}{16}\int_0^{\pi/3}\frac{1-\cos^2\theta}{\cos\theta}\sin\theta\mathrm{d}\theta$$

This is our usual trig integral where $u = \cos\theta \Rightarrow \mathrm{d}u = -\sin\theta\mathrm{d}\theta$,

$$I = -\frac{3}{16}\int_1^{1/2}\frac{1-u^2}{u^2}\mathrm{d}u = \frac{3}{16}\left[u+\frac{1}{u}\right]_1^{1/2} = \frac{3}{32}.$$

(6) $I = \int \frac{x\mathrm{d}x}{\sqrt{3-2x-x^2}}$.

Solution: This one is going to take a bit of ingenuity. Lets tinker with $3-2x-x^2 = 3-(x^2+2x)$. Notice we can get a perfect square if we add a 1 to x^2+2x, but if we add a 1 we must also "subtract" a 1, so $3-2x-x^2 = 3-(x^2+2x+1)+1 = 4-(x+1)^2$. Now, let $u = x+1 \Rightarrow \mathrm{d}u = \mathrm{d}x$, then

$$I = \int \frac{(u-1)\mathrm{d}u}{\sqrt{4-u^2}}.$$

This is precisely the form of the first case, so we substitute $u = 2\sin\theta \Rightarrow \mathrm{d}u = 2\cos\theta\mathrm{d}\theta$.

$$\int \frac{2\sin\theta-1}{\sqrt{4-4\sin^2\theta}}2\cos\theta\mathrm{d}\theta = \int\frac{2\sin\theta-1}{2\cos\theta}2\cos\theta\mathrm{d}\theta = -2\cos\theta-\theta+C$$

Since $u = 2\sin\theta$, $\sin\theta = u/2$, then $2\cos\theta = \sqrt{4-u^2}$, so

$$I = -\sqrt{4-u^2} - \sin^{-1}\left(\frac{u}{2}\right) + C = \sqrt{3-2x-x^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C.$$