8.4 INTEGRATION OF RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

Lets use the following example as motivation:

Ex: Consider $I = \int \frac{x+5}{x^2+x-2} \mathrm{d}x$.

Solution: Notice we can easily factor the denominator into $x^2 + x - 2 = (x-1)(x+2)$. Then we know that this looks like the common denominator of the sum of two fractions. Lets consider $\frac{1}{x-1} + \frac{1}{x+2} = \frac{2x+1}{(x-1)(x+2)}$. This is clearly not what we want, but this gives us an indication of the form of the fractions, namely

$$\frac{x+5}{x^2+x-2} = \frac{A}{x-1} + \frac{B}{x+2} = \frac{B(x-1)+A(x+2)}{(x-1)(x+2)} = \frac{(A+B)x+(2A-B)}{x^2+x-2}.$$

where A and B are some constants. Our task now is to solve for A and B. We notice that we must equate the numerators, i.e. $x + 5 = (A+B)x+(2A-B)$, so by matching the coefficients we get two equations: $A+B = 1$ and $2A-B = 5$. From the first equation we have $B = 1-A$. Then plugging B into the second equation gives $2A-1+A = 3A-1 = 5 \Rightarrow A = 2$. Then, $B = 1 - A = 1 - 2 = -1$. Now, we can plug these back into the fraction and put them back in the integral,

$$I = \int \frac{2\mathrm{d}x}{x-1} - \int \frac{\mathrm{d}x}{x+2} = 2\ln|x-1| - \ln|x+2| + C.$$

We digress slightly to do an example that does not involve partial fractions but that involves long division - a skill that will be very important for many of these types of problems,

Ex: $I = \int \frac{x^3+x}{x-1} \mathrm{d}x$.

Solution: By long division we get,

$$\frac{x^3+x}{x-1} = x^2 + x + 2 + \frac{2}{x-1}.$$

If you're having trouble with long division please come see me, asap! Then, putting this back into the integral gives,

$$\int \frac{x^3+x}{x-1} = \int \left(x^2 + x + 2 + \frac{2}{x-1}\right) \mathrm{d}x = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x + 2\ln|x-1| + C.$$

Whenever the highest power in the numerator is greater than or equal to the highest power in the denominator we must use long division. Once it's in a form we can use, we can go ahead and use partial fractions. We can split the types of problems we will come across on the exam into four cases detailed bellow.

From this point on we will consider integrals of the type:

$$\int f(x)\mathrm{d}x; \qquad f(x) = \frac{P(x)}{Q(x)}, \text{ where } P \text{ and } Q \text{ are polynomials.} \tag{1}$$

Case 1.

Suppose Q is a product of distinct linear factors, i.e. $Q = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$. Then,

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}. \tag{2}$$

(1) Convert $\frac{x^2+2x-1}{2x^3+3x^2-2x}$ into partial fractions.
Solution: First we factor out the denominator,

$$2x^3 + 3x^2 - 2x = x(2x-1)(x+2).$$

Then,

$$\begin{aligned}\frac{x^2+2x-1}{2x^3+3x^2-2x} &= \frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2} \\ &= \frac{A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)}{x(2x-1)(x+2)} \\ &= \frac{(2A+B+2C)x^2 + (3A+2B-C)x - 2A}{2x^3+3x^2-2x}.\end{aligned}$$

Now we equate the numerators to find our constants,

$$x^2 + 2x - 1 = (2A+B+2C)x^2 + (3A+2B-C)x - 2A.$$

Matching the coefficients give us the following equations,

$$\begin{aligned}2A + B + 2C &= 1 \\ 3A + 2B - C &= 2 \\ 2A &= 1\end{aligned}$$

The easiest one to solve for is $\boldsymbol{A = 1/2}$. Plugging this into the first equation gives, $B + 2C = 0 \Rightarrow B = -2C$. Plugging this into the second equation gives, $3/2 - 5C = 2 \Rightarrow -5C = 1/2 \Rightarrow \boldsymbol{C = -1/10} \Rightarrow \boldsymbol{B = 1/5}$.

(2) Convert $\frac{1}{x^2-a^2}$ into partial fractions.
Solution:

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a} = \frac{(A+B)x + (A-B)a}{x^2-a^2}.$$

Matching the coefficients gives us $A + B = 0 \Rightarrow A = -B$ straight away. Then we plug this into $(A-B)a = 2Aa = 1 \Rightarrow \boldsymbol{A = 1/2a} \Rightarrow \boldsymbol{B = -1/2a}$.

Case 2.

Suppose Q is a product of linear factors, some of which are repeated. Then, the repeated factors are of this form

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(ax+b)^r} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_r}{(ax+b)^r}. \tag{3}$$

(1) Convert $\frac{x^3-x+1}{x^2(x-1)^3}$ into partial fractions.
Solution: For this problem we simply put it into partial fractions form without finding the constants. Notice that the denominator is already in factored form.

$$\frac{x^3-x+1}{x^2(x-1)^3} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{B_1}{x-1} + \frac{B_2}{(x-1)^2} + \frac{B_3}{(x-1)^3}.$$

(2) Convert $\frac{x^4-2x^3+4x+1}{x^3-x^2-x+1}$ into partial fractions.
Solution: Notice that we must use long division because the highest power of the numerator is greater than the highest power of the denominator,

$$\frac{x^4-2x^3+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1}.$$

Now, we factor the denominator,

$$x^3-x^2-x+1 = x^2(x-1)-(x-1) = (x-1)(x^2-1) = (x-1)(x-1)(x+1) = (x-1)^2(x+1).$$

Then,

$$\begin{aligned}\frac{4x}{x^3-x^2-x+1} &= \frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}\\ &= \frac{A(x-1)(x+1)+B(x+1)+C(x-1)^2}{(x-1)^2(x+1)}\\ &= \frac{(A+C)x^2+(B-2C)x+(-A+B+C)}{x^3-x^2-x+1}.\end{aligned}$$

Equating the numerator gives,

$$4x = (A+C)x^2+(B-2C)x+(-A+B+C)$$

Matching the coefficients gives,

$$\begin{aligned} A+C &= 0\\ B-2C &= 4\\ -A+B+C &= 0.\end{aligned}$$

From the first equation we get $C=-A$, then plugging into the third equation gives $C+B+C = B+2C = 0 \Rightarrow B = -2C$. Plugging this into the second equation gives $-2C-2C = -4C = 4 \Rightarrow \boldsymbol{C=-1} \Rightarrow \boldsymbol{A=1} \Rightarrow \boldsymbol{B=2}$.

Case 3.

Suppose Q is a product of quadratic factors with no repeats, i.e. $Q = (a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x + c_2)\cdots(a_kx^2 + b_kx + c_k)$. Then,

$$\begin{aligned}\frac{P(x)}{Q(x)} &= \frac{P(x)}{(a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x + c_2)\cdots(a_kx^2 + b_kx + c_k)} \\ &= \frac{A_1x + B_1}{a_1x^2 + b_1x + c_1} + \frac{A_2x + B_2}{a_2x^2 + b_2x + c_2} + \cdots + \frac{A_kx + B_k}{a_kx^2 + b_kx + c_k}. \qquad (4)\end{aligned}$$

(1) Convert $\frac{x}{(x-2)(x^2+1)(x^2+4)}$ into partial fractions.
Solution: For this problem we simply put it into partial fractions form without finding the constants. Notice that the denominator is already in factored form.

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{x^2+4}.$$

(2) Convert $\frac{2x^2-x+4}{x^3+x}$ into partial fractions.
Solution: First we factor the denominator, $x^3 + x = x(x^2 + 1)$. Now, we put this into partial fractions form,

$$\frac{2x^2 - x + 4}{x^3 + x} = \frac{2x^2 - x + 4}{x(x^2+1)} = \frac{A}{x} + \frac{Bx + C}{x^2+1} = \frac{A(x^2+1) + Bx^2 + Cx}{x(x^2+1)} = \frac{(A+B)x^2 + Cx + A}{x^3 + x}.$$

Now, equating the numerators gives, $2x^2 - x + 4 = (A+B)x^2 + Cx + A$. We get that $\boldsymbol{A = 4}$ and $\boldsymbol{C = -1}$ straight away from matching the coefficients in front of x^1 and x^0. Now, from the x^2 coefficient we have $A + B = 4 + B = 2 \Rightarrow \boldsymbol{B = -2}$.

Case 4.

Suppose Q is product of factors that include repeated quadratic factors. Then the repeated quadratic factors will be of the form,

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(ax^2+bx+c)^r} = \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}. \tag{5}$$

(1) Convert $\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3}$ into partial fractions.
Solution: For this problem we simply put it into partial fractions form without finding the constants. Notice that the denominator is already in factored form.

$$\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} + \frac{E_1x+F_1}{x^2+1} + \frac{E_2x+F_2}{(x^2+1)^2} + \frac{E_3x+F_3}{(x^2+1)^3}.$$

(2) Convert $\frac{1-x+2x^2-x^3}{x(x^2+1)^2}$ into partial fractions.
Solution: Notice, the denominator is already factored, so we go right to it

$$\begin{aligned}\frac{1-x+2x^2-x^3}{x(x^2+1)^2} &= \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} = \frac{A(x^2+1)^2+(Bx+C)x(x^2+1)+Dx^2+Ex}{x(x^2+1)^2} \\ &= \frac{(A+B)x^4+Cx^3+(2A+B+D)x^2+(C+E)x+A}{x(x^2+1)^2}\end{aligned}$$

Equating the numerators gives,

$$1-x+2x^2-x^3 = (A+B)x^4+Cx^3+(2A+B+D)x^2+(C+E)x+A.$$

We get $\boldsymbol{A=1}$ for free, and from that we get $A+B=1+B=0 \Rightarrow \boldsymbol{B=-1}$. We also get $\boldsymbol{C=-1}$ for free, from which we get $C+E=-1+E=-1 \Rightarrow \boldsymbol{E=0}$. And finally we get $2A+B+D=2-1+D=1+D=2 \Rightarrow \boldsymbol{D=1}$.

Problems not on the exam, but are important nonetheless.

(1) Solve $\int \frac{4x^2-3x+2}{4x^2-4x+3}\mathrm{d}x$.
Solution: First and foremost we must use long division because the highest power in the numerator is equal to the highest power in the denominator. After doing long division we get,

$$\frac{4x^2-3x+2}{4x^2-4x+3} = 1 + \frac{x-1}{4x^2-4x+3}.$$

Notice, $4x^2-4x+3$ can not be factored because it's discriminant is, $b^2-4ac = 4^2-4(4)(3) = -32 < 0$ (i.e. the thing under the radical in the quadratic formula). Recall that we can only factor quadratic polynomials whose discriminant is greater than or equal to zero.

It's easy to integrate 1, so lets focus on integrating the second part, but lets try to put it into a form that will allow us to use a u-sub.

$$I = \int \frac{x-1}{4x^2-4x+3}\mathrm{d}x = \int \frac{x-1}{(2x-1)^2+2}.$$

We get this by completing the square on $4x^2-4x$, that is

$$4x^2-4x = 4(x^2-x) = 4\left(x^2-x+\frac{1}{4}\right)-1 = 4\left(x^2-\frac{1}{2}\right)^2-1 = (2x^2-1)-1.$$

Now we use u-sub where $u = 2x-1 \Rightarrow \mathrm{d}u = 2\mathrm{d}x$. Then,

$$I = \frac{1}{4}\int \frac{u-1}{u^2+2}\mathrm{d}u = \frac{1}{4}\int \frac{u}{u^2+2}\mathrm{d}u - \frac{1}{4}\int \frac{1}{u^2+2}\mathrm{d}u$$

We know how to solve both integrals. In case you don't, you must start coming to office hours.

(2) Solve $I = \int \frac{\sqrt{x+4}}{x}\mathrm{d}x$.
Solution: Lets use the u-sub, $u^2 = x+4 \Rightarrow 2u\mathrm{d}u = \mathrm{d}x$, plugging this in gives,

$$I = 2\int \frac{u^2\mathrm{d}u}{u^2-4}.$$

Now, we must use long division to get,

$$I = 2\int \left(1+\frac{4}{u^2-4}\right)\mathrm{d}u = 2u + 2\int \frac{4}{(u-2)(u+2)}.$$

Now, we split $4/(u-2)(u+2)$ into partial fractions,

$$\frac{4}{(u-2)(u+2)} = \frac{A}{u-2} + \frac{B}{u+2} = \frac{A(u+2)+B(u-2)}{(u-2)(u+2)} = \frac{(A+B)u+2(A-B)}{(u-2)(u+2)}.$$

Equating the numerators gives $4 = (A+B)u + 2(A-B)$, then we have that $A+B = 0 \to B = -A$. We plug this into the second term to get, $2(A-B) = 2(A+A) = 4A = 4 \Rightarrow \boldsymbol{A=1} \Rightarrow \boldsymbol{B=-1}$. After this point we know how to solve the two integrals.

8.6 Numerical Integration

We derived three methods of solving integrals numerically in class.

Midpoint rule:

$$\int_a^b f(x)\mathrm{d}x \approx \Delta x[f(x_1^*)+f(x_2^*)+\cdots+f(x_n^*)];\ x_i^* = \frac{1}{2}(x_i+x_{i+1}),\ \Delta x = \frac{b-a}{n} \tag{6}$$

Where n is the number of intervals or equivalently the number of "steps".

$$\text{Error bound: } |E_M| \leq \frac{K(b-a)^3}{24n^2};\ |f''(\xi)| \leq K,\ \xi \in [a,b]. \tag{7}$$

Where $|f''(\xi)|$ is just the maximum of the second derivative in $[a,b]$.

Ex: Consider the integral $I = \int_1^2 \frac{\mathrm{d}x}{x}$. We note that the exact value of this integral is $I = \ln 2 \approx .693147$.

(a) Approximate the integral via Midpoint rule with $n = 5$ steps.
Solution: Here $a = 1$, $b = 2$, so $\Delta x = 1/5$. Also, clearly $x_i = a+i\Delta x$, so $x_0 = a = 1$, $x_1 = 1.2$, $x_2 = 1.4$, $x_3 = 1.6$, $x_4 = 1.8$, and $x_5 = b = 2$, so $x_1^* = 1.1$, $x_2^* = 1.3$, $x_3^* = 1.5$, $x_4^* = 1.7$, $x_5^* = 1.9$ Then plugging this into the formula gives,

$$I \approx \frac{1}{5}\left[\frac{1}{1.1}+\frac{1}{1.3}+\frac{1}{1.5}+\frac{1}{1.7}+\frac{1}{1.9}\right] \approx .691908.$$

(b) Find the error bound for this approximation.
Solution: Notice that $b-a = 1$ and $n = 5$. Now, we must just find K. To do this we take the second derivative $f''(x) = 2/x^3$. We notice that in $[1,2]$ this is greatest at $\xi = 1$, so $f''(\xi) = 2$. So, we choose $K = 2$. Plugging these into the formula gives,

$$|E_M| \leq \frac{K(b-a)^3}{24n^2} = \frac{2\cdot 1}{24\cdot 25} = \frac{1}{300}.$$

(c) Find the smallest n that guarantees $|E_M| \leq .0001$.
Solution: This is a far more interesting problem. We start with the formula and put in the quantities we know,

$$\frac{K(b-a)^3}{24n^2} = \frac{1}{12n^2} \leq .0001 \Rightarrow n^2 \geq \frac{1}{.0012} \Rightarrow n \geq \frac{1}{\sqrt{.0012}} \approx 28.8$$

This gives us $n = 29$.

Trapezoid rule:

$$\int_a^b f(x)\mathrm{d}x \approx \frac{\Delta x}{2}[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)]; \tag{8}$$

$$\Delta x = \frac{b-a}{n},\ x_i = a+i\Delta x.$$

Where n is the number of intervals or equivalently the number of "steps".

$$\text{Error bound: } |E_T| \leq \frac{K(b-a)^3}{12n^2};\ |f''(\xi)| \leq K,\ \xi \in [a,b]. \tag{9}$$

Where $|f''(\xi)|$ is just the maximum of the second derivative in $[a,b]$.

Ex : Take the same integral as in the Midpoint rule example, and answer the same exact questions.

(a) We already have the quantities we need from the Midpoint rule example, so we just plug those quantities into the Trapezoid rule formula,

$$I \approx \frac{1}{10}\left[1 + 2\frac{1}{1.2} + 2\frac{1}{1.4} + 2\frac{1}{1.6} + 2\frac{1}{1.8} + \frac{1}{2}\right] \approx .695635.$$

(b) For the error bound the difference between trapezoid rule and midpoint rule is a factor of 2, so plugging into the formula gives $|E_T| \leq 1/150$.

(c) We have the same quantities here as in the midpoint rule problem, so we get $n > 40.8 \Rightarrow n = 41$.

Simpson's rule:

$$\int_a^b f(x)\mathrm{d}x \approx \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)]; \tag{10}$$

$$\Delta x = \frac{b-a}{n},\ n \geq 4\text{and must be even.}$$

Where n is the number of intervals or equivalently the number of "steps".

$$\text{Error bound: } |E_S| \leq \frac{K(b-a)^5}{180n^4};\ |f^{(4)}(\xi)| \leq K,\ \xi \in [a,b]. \tag{11}$$

Where $|f^{(4)}(\xi)|$ is just the maximum of the fourth derivative in $[a,b]$.

Ex : Consider the same integral as the previous two examples.

(a) Approximate this integral with $n = 10$ steps.
Solution: Here, $\Delta x = 1/10$, and $x_0 = 1$, $x_1 = 1.1$, $x_2 = 1.2$, $x_3 = 1.3$, $x_4 = 1.4$, $x_5 = 1.5$, $x_6 = 1.6$, $x_7 = 1.7$, $x_8 = 1.8$, $x_9 = 1.9$, and $x_10 = 2$. Plugging these into the formula gives,

$$I \approx \frac{1}{30}\left[1 + 4\frac{1}{1.1} + 2\frac{1}{1.2} + 4\frac{1}{1.3} + 2\frac{1}{1.4} + 4\frac{1}{1.5} + 2\frac{1}{1.6} + 4\frac{1}{1.7} + 2\frac{1}{1.8} + 4\frac{1}{1.9} + \frac{1}{2}\right] \approx .693150.$$

(b) Find the smallest n that guarantees $|E_S| \leq .0001$.
Solution: We have most of the quantities, so we must only look for K. Taking the fourth derivative gives $f^{(4)}(x) = 24/x^5$. We see that this is greatest at $\xi = 1$ for our interval, so $f^{(4)}(\xi) = 24$, hence we choose $K = 24$. Plugging these into the formula gives

$$|E_S| \leq \frac{24}{180n^4} \leq .0001 \Rightarrow n^4 \geq \frac{24}{180(.0001)} \Rightarrow n \geq \left(\frac{24}{180(.0001)}\right)^{1/4} \approx 6.04.$$

So, we have $n = 8$ because we need an even n.