8.7 Improper Integrals

Improper integrals are integrals that may blow up. This poses a question, what is infinity and how do we deal with it? Consider the following example

Ex: $\int_1^\infty \mathrm{d}x/x^2$. We know how to integrate this for a finite interval, so why don't we do that and then take the infinite limit.

$$\int_1^\infty \frac{\mathrm{d}x}{x^2} = \lim_{t\to\infty}\int_1^t \frac{\mathrm{d}x}{x^2} = \lim_{t\to\infty} \frac{-1}{x}\Big|_1^t = \lim_{t\to\infty} 1 - \frac{1}{t} = 1.$$

We have two cases of improper integrals. One where the interval is infinite and another where the interval is finite but the integrand has a discontinuity.

Case 1: Infinite Intervals

a) If $\displaystyle\int_a^t f(x)\mathrm{d}x$ exists for all $t \geq a$, then $\displaystyle\int_a^\infty f(x)\mathrm{d}x = \lim_{t\to\infty}\int_a^t f(x)\mathrm{d}x$.

b) If $\displaystyle\int_t^b f(x)\mathrm{d}x$ exists for all $t \leq b$, then $\displaystyle\int_{-\infty}^b f(x)\mathrm{d}x = \lim_{t\to-\infty}\int_t^b f(x)\mathrm{d}x$.

Definition 1. If $\int_a^\infty f(x)\mathrm{d}x$ and $\int_{-\infty}^b f(x)\mathrm{d}x$ are convergent if the limit exists, and divergent if the limit does not exist.

$$c)\text{If } \int_a^\infty f(x)\mathrm{d}x and \int_{-\infty}^a f(x)\mathrm{d}x \text{ are convergent ,}$$
$$\int_{-\infty}^\infty f(x)\mathrm{d}x = \int_{-\infty}^a f(x)\mathrm{d}x + \int_a^\infty f(x)\mathrm{d}x.$$

Determine whether the following integrals are convergent or divergent:

(1) $\int_1^\infty \mathrm{d}x/x$.
Solution: Following the same procedure as the above example gives,

$$\int_1^\infty \frac{\mathrm{d}x}{x} = \lim_{t\to\infty}\int_1^t \frac{\mathrm{d}x}{x} = \lim_{t\to\infty} \ln|x|\big|_1^t = \lim_{t\to\infty} \ln|t| = \infty$$

(2) $\int_{-\infty}^0 xe^x\mathrm{d}x$.
Solution: Notice that we must integrate this by parts with $u = x \Rightarrow \mathrm{d}u = \mathrm{d}x$ and $\mathrm{d}v = e^x\mathrm{d}x \Rightarrow v = e^x$.

$$\int_{-\infty}^0 xe^x\mathrm{d}x = \lim_{t\to-\infty}\int_t^0 xe^x\mathrm{d}x = \lim_{t\to-\infty} xe^x|_t^0 - \int_t^0 e^x\mathrm{d}x = \lim_{t\to-\infty} xe^x|_t^0 - e^x|_t^0 = \lim_{t\to-\infty} -te^t - 1 + e^t.$$

For the first limit we need to employ L'Hôpital's rule,

$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty}\frac{t}{e^{-t}} = \lim_{t\to-\infty}\frac{1}{-e^{-t}} = \lim_{t\to-\infty} -e^t = 0.$$

Therefore, $\int_{-\infty}^0 xe^x\mathrm{d}x = -1$

(3) $\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^2}$.
Solution: Here we need to split the integral in two. The easiest way to split it is right down the middle,

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^2} = \int_{-\infty}^{0} \frac{\mathrm{d}x}{1+x^2} + \int_{0}^{\infty} \frac{\mathrm{d}x}{1+x^2}$$

Lets call the first integral I_1 and the second integral I_2. We must integrate these separately,

$$I_1 = \lim_{t\to-\infty} \int_t^0 \frac{\mathrm{d}x}{1+x^2} = \lim_{t\to-\infty} \tan^{-1} x|_t^0 = \lim_{t\to-\infty} -\tan^{-1} t = \frac{\pi}{2}$$
$$I_2 = \lim_{t\to\infty} \int_0^t \frac{\mathrm{d}x}{1+x^2} = \lim_{t\to\infty} \tan^{-1} x|_0^t = \lim_{t\to\infty} \tan^{-1} t = \frac{\pi}{2}$$

Then, $I = I_1 + I_2 = \pi$.

Lets look at this very special example,

Ex: For what values of p is the integral $\int_1^\infty \mathrm{d}x/x^p$?
Solution: Lets assume $p \neq 1$, since that case is slightly different, and we have also dealt with that case in a previous example where it was divergent. First lets integrate and then deal with the two cases when we take the limit.

$$\int_1^\infty \frac{\mathrm{d}x}{x^p} = \lim_{t\to\infty} \int_1^t x^{-p}\mathrm{d}x = \lim_{t\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^t = \lim_{t\to\infty} \frac{1}{1-p}\left[\frac{1}{t^{p-1}} - 1\right].$$

If $p > 1$, $p-1 > 0$, then as $t \to \infty$, $t^{p-1} \to \infty$, so $1/t^{p-1} \to 0$, therefore $\int_1^\infty \frac{\mathrm{d}x}{x^p} = \frac{1}{p-1}$, and hence it converges.
If $p < 1$, $p-1 < 0$, so as $t \to \infty$, $1/t^{p-1} = t^{1-p} \to \infty$, therefore the integral diverges. And we know for $p = 1$, the integral diverges as well.

We have just proved a theorem, which we state bellow

Theorem 1. *Consider $\int_1^\infty \frac{dx}{x^p}$. If $p > 1$, the integral converges to $\frac{1}{p-1}$, otherwise it diverges.*

Now, we move on to the second case, which is the case of finite intervals where the integrand has a discontinuity.

Case 2: Integrands with Discontinuities.

a) If f is continuous in $[a,b)$ and discontinuous at $x=b$, then $\displaystyle\int_a^b f(x)\mathrm{d}x = \lim_{t\to b^-}\int_a^t f(x)\mathrm{d}x.$

b) If f is continuous in $(a,b]$ and discontinuous at $x=a$, then $\displaystyle\int_a^b f(x)\mathrm{d}x = \lim_{t\to a^+}\int_t^b f(x)\mathrm{d}x.$

Definition 2. The integral $\int_a^b f(x)\mathrm{d}x$ is said to be convergent if the limit exists, and divergent if the limit does not exist.

c) If f has a discontinuity at $c\in[a,b]$ and $\displaystyle\int_a^c f(x)\mathrm{d}x$ and $\displaystyle\int_c^b f(x)\mathrm{d}x$ both converge, then

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x.$$

Determine whether the following integrals are convergent or divergent:

(1) $\int_2^5 \frac{\mathrm{d}x}{\sqrt{x-2}}$.
Solution: Our discontinuity is at $x=2$, so

$$\int_2^5 \frac{\mathrm{d}x}{\sqrt{x-2}} = \lim_{t\to 2^+}\int_t^5 \frac{\mathrm{d}x}{\sqrt{x-2}} = \lim_{t\to 2^+} 2\sqrt{x-2}\Big|_t^5 = \lim_{t\to 2^+} 2(\sqrt{3}-\sqrt{t-2}) = 2\sqrt{3}.$$

(2) $\int_0^{\pi/2}\sec x\mathrm{d}x$.
Solution: The discontinuity is at $x=\pi/2$, so

$$\int_0^{\pi/2}\sec x\mathrm{d}x = \lim_{t\to(\pi/2)^-}\int_0^t \sec x\mathrm{d}x = \lim_{t\to(\pi/2)^-}\ln|\sec x+\tan x|\Big|_0^t = \lim_{t\to(\pi/2)^-}\ln|\sec t+\tan t| = \infty.$$

Therefore, it diverges. Notice that we didn't have to use L'Hôpital's rule because both $\sec x$ and $\tan x$ blow up in the same direction, whereas if they blew up in opposite directions we would have to use L'Hôpital's rule.

(3) $\int_0^3 \frac{\mathrm{d}x}{x-1}$.
Solution: Here we have a discontinuity at $x=1$, so we must break the integral up, $\int_0^3 \frac{\mathrm{d}x}{x-1} = \int_0^1 \frac{\mathrm{d}x}{x-1} + \int_1^3 \frac{\mathrm{d}x}{x-1}$. Lets integrate the first integral,

$$\int_0^1 \frac{\mathrm{d}x}{x-1} = \lim_{t\to 1^-}\int_0^t \frac{\mathrm{d}x}{x-1} = \lim_{t\to 1^-}\ln|x-1|\Big|_0^t = \lim_{t\to 1^-}(\ln|t-1|) = -\infty.$$

Since one of the integrals diverge the entire integral diverges. Notice, if we integrated without breaking the integral up we would have gotten a false conclusion. Try it out for yourself.

(4) $\int_0^1 \ln x \mathrm{d}x$.
Solution: Here the discontinuity is at $x = 1$, and we also must integrate by parts with $u = \ln x \Rightarrow \mathrm{d}u = 1/x$ and $\mathrm{d}v = \mathrm{d}x \Rightarrow v = x$,

$$\int_0^1 \ln x \mathrm{d}x = \lim_{t\to 0^+} \int_t^1 \ln x \mathrm{d}x = \lim_{t\to 0^+} x \ln x\big|_t^1 - \int_t^1 \mathrm{d}x = \lim_{t\to 0^+} -t\ln t - 1 + t.$$

We deal with the first limit using L'Hôpital's rule,

$$\lim_{t\to 0^+} t \ln t = \lim_{t\to 0^+} \frac{\ln t}{1/t} = \lim_{t\to 0^+} \frac{1/t}{-1/t^2} = \lim_{t\to 0^+} -t = 0.$$

Therefore, the integral is convergent and $\int_0^1 \ln x \mathrm{d}x = -1$.

Comparison Tests.

Many times we may not be able to evaluate an integral or an integral may be too difficult to evaluate in a reasonable time frame, but we would still like to know the behavior of the integral, which translates to the behavior of certain differential equations.

Theorem 2. *Direct comparison test: If f, g are continuous with $f \geq g \geq 0$ for $x \geq a$,*

$$a)\ \int_a^\infty f(x)dx \text{ converges } \Rightarrow \int_a^\infty g(x)dx \text{ converges.}$$
$$b)\ \int_a^\infty g(x)dx \text{ diverges } \Rightarrow \int_a^\infty f(x)dx \text{ diverges.}$$

For the following examples, state whether or not the integral converges or diverges, and explain why.

(1) $\int_0^\infty e^{-x^2} \mathrm{d}x$.
Solution: We can't evaluate this directly with the methods we have learned thus far, so we must use a comparison test. Notice, for $x \geq 1$, $e^{-x} \geq e^{-x^2}$, and we can prove that $\int_1^\infty e^{-x}$ converges. Now, if we weren't sure about the convergence we could go ahead and use the limit comparison test. To prove that this converges we just take the integral,

$$\int_1^\infty e^{-x} \mathrm{d}x = \lim_{t\to\infty} \int_1^t e^{-x} \mathrm{d}x = \lim_{t\to\infty} (e^{-1} - e^{-t}) = e^{-1}.$$

Therefore, by the direct comparison test, $\int_0^\infty e^{-x^2} \mathrm{d}x$ converges.

(2) $\int_1^\infty [(1 + e^{-x})/x] \mathrm{d}x$.
Solution: Notice, this integral would be a pain to evaluate, but we get a feeling that it diverges. Now, $\frac{1+e^{-x}}{x} \geq \frac{1}{x}$ because the exponential function is always positive. Now, we know that $\int_1^\infty \mathrm{d}x/x$ diverges because $p = 1$. Therefore, by the direct comparison test $\int_1^\infty [(1 + e^{-x})/x] \mathrm{d}x$ diverges.

Theorem 3. *Limit comparison test: If f, g are continuous and $\lim_{x\to\infty} f(x)/g(x) = L$, then either, $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$, both converge or both diverge.*

For the following examples, state whether or not the integral converges or diverges, and explain why.

(1) $\int_1^\infty \frac{\sqrt{x^2+1}}{x^3}\mathrm{d}x$.
Solution: Here the root gives us some difficulty in finding a direct comparison, but we can find something that would work for limit comparison. Since the problem is in the numerator, lets divide through by the highest power of the numerator,

$$\frac{\sqrt{x^2+1}}{x^3} = \frac{\sqrt{1+1/x^2}}{x^2} \sim \frac{1}{x^2}.$$

Now we must take the limit of the ratios to prove that this is a valid comparison,

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}/x^3}{1/x^2} = \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} = \lim_{x\to\infty} \sqrt{1+\frac{1}{x^2}} = 1.$$

Since this is a valid comparison, and we know that $\int_1^\infty \mathrm{d}x/x^2$ converges because $p > 1$, by the limit comparison test $\int_1^\infty \frac{\sqrt{x^2+1}}{x^3}\mathrm{d}x$ also converges.

(2) $\int_1^\infty \frac{x}{\sqrt{x^3+2}}\mathrm{d}x$.
Solution: Here the problem is with the denominator, so lets divide through by the highest power of the denominator,

$$\frac{x}{\sqrt{x^3+2}} = \frac{x/\sqrt{x^3}}{\sqrt{x^3+2}/\sqrt{x^3}} = \frac{1/\sqrt{x}}{\sqrt{1+2/x^3}} \sim \frac{1}{\sqrt{x}}.$$

Now we take the limit of the ratios,

$$\lim_{x\to\infty} \frac{x/\sqrt{x^3+2}}{1/\sqrt{x}} = \lim_{x\to\infty} \frac{\sqrt{x^3}}{\sqrt{x^3+2}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+2/x^3}} = 1.$$

Now, $\int_1^\infty \mathrm{d}x/\sqrt{x}$ diverges because $p < 1$, so by the limit comparison test, $\int_1^\infty \frac{x}{\sqrt{x^3+2}}\mathrm{d}x$.

10.1 Sequences

Sequences are just functions, except as opposed to standard functions whose domains are the real numbers, the domain for sequences are the integers. So, we can think of them as regular functions, but we must be careful in certain instances.

Lets quickly go through a few different ways of representing a sequence,

a) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$; $\quad a_n = \frac{n}{n+1}$; $\quad \left\{\frac{1}{2}, \frac{2}{3}, \cdots, \frac{n}{n+1}, \cdots\right\}$

b) $\left\{\frac{(-1)^n(n+1)}{3^n}\right\}_{n=1}^{\infty}$; $\quad a_n = \frac{(-1)^n(n+1)}{3^n}$; $\quad \left\{-\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \cdots, \frac{(-1)^n(n+1)}{3^n}, \cdots\right\}$

c) $\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}$; $\quad a_n = \sqrt{n-3}, n \geq 3$; $\quad \{0, 1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n-3}, \ldots\}$

d) $\left\{\cos\frac{n\pi}{6}\right\}_{n=0}^{\infty}$; $\quad a_n = \cos\frac{n\pi}{6}, n \geq 0$ $\quad \left\{1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \ldots, \cos\frac{n\pi}{6}, \cdots\right\}$

An important skill to have is deriving a general formula for a sequence from looking at a few terms of the sequence,

Ex: Find a formula for $\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \cdots\right\}$.

Solution: The first thing we notice is that there is an alternating sign, and since the first element (n=1) is positive, we need to start of with an even power of -1, so $(-1)^{n-1}$ works. Notice we could have also used $(-1)^{n+1}$. We also notice that the denominators are respective powers of 5, so the denominator must be 5^n. Now, we notice that the numerator starts with 3 and goes up by one ever time, so the numerator is $n+2$, then $a_n = (-1)^{n-1}\frac{n+2}{5^n}$.

We can also take limits of sequences, which is what we are most interested in for this class,

Ex: Lets take limits of the sequences we've seen today,

$$\text{a)} \lim_{n\to\infty}\frac{n}{n+1} = 1 \qquad \text{b)} \lim_{n\to\infty}(-1)^n\frac{n+1}{3^n} = 0 \qquad \text{c)} \lim_{n\to\infty}\sqrt{n-3} = \infty$$

$$\text{d)} \lim_{n\to\infty}\cos\frac{n\pi}{6} \text{ DNE} \qquad \text{e)} \lim_{n\to\infty}(-1)^{n-1}\frac{n+2}{5^n} = 0.$$

Just as with standard functions we can define convergence and divergence,

Definition 3. If $\lim_{n\to\infty} a_n = L$ we say it is convergent, otherwise it is divergent.

Lets remind ourselves of the standard limit laws,

Theorem 4. *If $\{a_n\}$ and $\{b_n\}$ are convergent sequences, then*

$$a)\ \lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n \qquad b)\ \lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n \qquad c)\ \lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$$

$$d)\ \lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \text{ if } \lim_{n\to\infty} b_n \neq 0 \qquad e)\ \lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \text{ if } p > 0 \text{ and } a_n \geq 0$$

We also have the squeeze theorem and a very important consequence of the squeeze theorem,

Theorem 5. *If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.*

Theorem 6. *If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.*

(1) Let's do a few more easy example before getting to the tough ones,

$$\text{a)} \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0 \qquad \text{b)} \lim_{n\to\infty} (-1)^n \text{ DNE} \qquad \text{c)} \lim_{n\to\infty} \frac{(-1)^n}{n} = 0.$$

(2) Does $a_n = \frac{n!}{n^n}$ diverge or converge?
Solution: This is a tough one, and we may not be able to see what is going on right away, so it's a good idea to write down the first few terms for the n^{th} element of the sequence, $a_n = \frac{1\cdot 2\cdot 3\cdot\cdots\cdot n}{n\cdot n\cdot n\cdot\cdots\cdot n}$. Now, we get a better idea of what's going on. Lets factor out $1/n$, since we know what happens to that sequence. If we do this, we notice $\frac{2\cdot 3\cdot\cdots\cdot n}{n\cdot n\cdot\cdots\cdot n} \leq 1$ for $n \geq 1$. Then our sequence is always positive, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdot\cdots\cdot n}{n\cdot n\cdot n\cdot\cdots\cdot n} = \lim_{n\to\infty} \frac{1}{n}\left(\frac{2\cdot 3\cdot\cdots\cdot n}{n\cdot n\cdot\cdots\cdot n}\right) \leq \lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore, our sequence too, converges to 0.

(3) For what values of r does the sequence $a_n = r^n$ converge?
Solution: Lets take some different cases. If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$. If $|r| > 1$, then $\lim_{n\to\infty} r^n = \infty$. If $r = 1$, $\lim_{n\to\infty} r^n = 1$. Finally, if $r = -1$, $\lim_{n\to\infty} r^n$ does not exist. So, it converges for $|r| < 1$ and $r = 1$.

Monotonic Sequences.

What if we couldn't take the limit of a sequence, but we knew some things about the function and wanted to analyze the behavior. The following definitions and theorem will help us deal with this.

Definition 4. A sequence a_n is called nondecreasing(think increasing) if $a_n \leq a_{n+1}$ for all $n \geq 1$, i.e. $a_1 \leq a_2 \leq a_3 \leq \cdots$. It is called nonincreasing (think decreasing) if $a_n \geq a_{n+1}$ for all $n \geq 1$, i.e. $a_1 \geq a_2 \geq a_3 \geq \cdots$. These types of sequences are collectively called monotonic sequences.

(1) Is $3/(n+5)$ increasing or decreasing?
Solution: For this case it's easiest to compare the n^{th} term and the $(n+1)^{\text{th}}$ term. To do this we simply plug in and we notice,

$$\frac{3}{n+5} > \frac{3}{(n+1)+5} = \frac{3}{n+6}.$$

Since this is true for all $n \geq 1$, the sequence is decreasing.

(2) Show $\left\{\frac{n}{n^2+1}\right\}_{n=1}^{\infty}$ is decreasing.
Solution: For this it's easier to take the derivative,

$$\left(\frac{n}{n^2+1}\right)' = \frac{1-n^2}{(n^2+1)^2} < 0; \text{ for } n > 1.$$

Therefore, the sequence is decreasing.

Definition 5. A sequence a_n is said to be bounded above if there is an M such that $a_n \leq M$ for all $n \geq 1$, and bounded below if there is an m such that $a_n \geq m$ for all $n \geq 1$.

Theorem 7. *Every bounded monotonic sequence is convergent.*

Difference Equations (aka Recurrence Relations, aka Recursive Formula).

These types of sequences come up often in various applications. The idea is that subsequent elements in the sequence will depend on previous elements in the sequence. We can think of the $(n+1)^{\text{th}}$ term as a function of a combination of other terms, i.e. $a_{n+1} = f(a_n, a_{n-1}, \ldots, a_1)$.

(1) Lets try to find the limit of the following difference equation: $a_{n+1} = (a_n+6)/2$. Notice that if the limit exists, $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = a_*$. Here, a_* is called the fixed point of the difference equation. Now, we can plug this in and find the value for it,

$$a_* = \frac{1}{2}(a_* + 6) \Rightarrow a_* = 6.$$

Examples From the Book.

46) Notice $\lim_{n\to\infty} \left|\frac{\sin^2 n}{2^n}\right| \le \lim_{n\to\infty} \frac{1}{2^n} = 0 \Rightarrow \lim_{n\to\infty} \frac{\sin^2 n}{2^n} = 0.$

47) $\lim_{n\to\infty} \frac{n}{2^n} = \lim_{n\to\infty} \frac{n}{e^{n\ln 2}} = \lim_{n\to\infty} \frac{1}{\ln 2 e^{n\ln 2}} = 0.$

68) $\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\ln(1+1/n)}{1/n} = \lim_{n\to\infty} \frac{-1/(n^2+n)}{-1/n^2}$
$= \lim_{n\to\infty} \frac{n^2}{n^2+n} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1.$

71) For this problem we must take $e^{\ln}$, then take the limit

$$\lim_{n\to\infty}\left(\frac{x^n}{2n+1}\right)^{1/n} = \lim_{n\to\infty} e^{\ln(x^n/(2n+1))^n} = e^{\lim_{n\to\infty}\ln(x^n/(2n+1))/n}.$$

Lets first compute the limit then plug it back in,

$$\lim_{n\to\infty}\frac{\ln(x^n/(2n+1))}{n} = \lim_{n\to\infty}\frac{\ln(x^n)-\ln(2n+1)}{n} = \lim_{n\to\infty}\frac{n\ln x-\ln(2n+1)}{n}$$
$$= \lim_{n\to\infty}\frac{\ln x - 2/(2n+1)}{1} = \lim_{n\to\infty}\ln x - \frac{2}{2n+1} = \ln x.$$

Plugging this back in gives, $e^{\ln x} = x$.

72) For this we must use our $e^{\ln}$ trick again,

$$\lim_{n\to\infty}\left(1-\frac{1}{n^2}\right)^n = \lim_{n\to\infty} e^{\ln(1-1/n^2)^n} = e^{\lim_{n\to\infty} n\ln(1-1/n^2)}.$$

So, lets look at the limit then plug it back in,

$$\lim_{n\to\infty}\frac{\ln(1-1/n^2)}{1/n} = \lim_{n\to\infty}\frac{-2/(n-n^3)}{-1/n^2} = \lim_{n\to\infty}\frac{2n^2}{n-n^3} = \lim_{n\to\infty}\frac{2}{1/n-n} = 0.$$

Then, plugging back in gives, $e^0 = 1$.

84) Once again,

$$\lim_{n\to\infty} e^{\ln(n^2+n)^{1/n}} = e^{\lim_{n\to\infty} \frac{1}{n}\ln(n^2+n)}.$$

Computing the limit gives,

$$\lim_{n\to\infty} \frac{\ln(n^2+n)}{n} = \lim_{n\to\infty} \frac{(2n+1)/(n^2+n)}{1} = \lim_{n\to\infty} \frac{2n+1}{n^2+n} = \lim_{n\to\infty} \frac{2+1/n}{n+1} = 0.$$

Plugging back in gives, $e^0 = 1$.

90) We have done a problem like this in the improper integrals section. This reiterates the intimate relationship between integrals and sequences. $\lim_{n\to\infty} \int_1^n \mathrm{d}x/x^p$, for $p > 1$ converges to $1/(p-1)$. We can see this by integrating it.