10.2 SERIES

A series is a sum of sequential terms. An infinite series can be represented as such: $\sum_{n=1}^{\infty} a_n$. We also think of series as a sequence of partial sums, where each partial sum is $s_N = \sum_{n=1}^{N} a_n$. We have to make sure we don't confuse these very different sequences. One is a sequence that is being summed, the other is a sequence of sums.

Definition 1. Given $\sum_{n=1}^{\infty} a_n$, let $s_n = \sum_{i=1}^{n} a_i$ bet the partial sums. If s_n converges and $\lim_{n\to\infty} s_n = s$ exists, then we say $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} a_n = s$. Otherwise, we say it diverges.

Ex: Consider the series $\sum_{n=1}^{\infty} ar^{n-1}$. This is a very important series called the geometric series. What does this converge to?

Notice if $r = 1$, $s_n = a + a + \cdots + a = na \to \pm\infty$, so it diverges. Now, if $r = -1$, the partial sum will jump between zero and one, so it also diverges.

If $|r| \neq 1$, $s_n = a + ar + ar^2 + \cdots + ar^{n-1}$, and $rs_n = ar + ar^2 + \cdots + ar^n$, then $s_n - rs_n = a - ar^n \Rightarrow s_n = \frac{a(1-r^n)}{1-r}$. Now, for $-1 < r < 1$, $r^n \to 0$ as $n \to \infty$, hence $\lim_{n\to\infty} s_n = a/(1-r)$. For $|r| > 1$, $r^n \to \infty$, so s_n clearly diverges

Theorem 1. *The geometric series* $\sum_{n=1}^{\infty} ar^{n-1}$ *converges, for* $|r| < 1$ *to*

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \tag{1}$$

and diverges otherwise.

(1) Find the sum of $S = 5 - 10/3 + 20/9 - 40/27 + \cdots$.
Solution: Notice that we can immediately factor out a 5, $S = 5[1 - 2/3 + 4/9 - 8/27 + \cdots]$. Now we notice that we have alternating sings, so we must have a $(-1)^{n-1}$ because the first term is positive (if the first term was negative it would be $(-1)^n$). Next, we notice that all the terms are powers of 2/3, via the geometric series theorem, our sum is

$$\sum_{n=1}^{\infty} 5\left(-\frac{2}{3}\right)^{n-1} = \frac{5}{1+2/3} = \frac{5}{5/3} = 3.$$

(2) Is $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent or divergent?
Solution: This series isn't in the form of the geometric series, so we must convert it to that form,

$$\sum_{n=1}^{\infty} 2^{2n}3^{1-n} = \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}.$$

This does not converge because $4/3 > 1$, so it violates the hypothesis of the geometric series theorem.

(3) Write $2.3\overline{17}$ as a geometric series.
Solution: We must think of this as a constant plus a fraction,

$$2.3\overline{17} = 2.3+\frac{17}{10^3}+\frac{17}{10^5}+\cdots = 2.3+17\left[\frac{1}{10^3}+\frac{1}{10^5}\cdots\right] = 2.3+17\sum_{n=1}^{\infty}\left(\frac{1}{10}\right)^{2n+1}.$$

(4) For what values of x does $\sum_{n=0}^{\infty} x^n$ (this is called a power series) converge?
Solution: This is exactly a geometric series if x were fixed. Now, we may not be able to see this right away, but if we play around with the index we see that

$$\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x};\ |x| < 1. \tag{2}$$

The next couple of examples are telescoping and harmonic series. These will illustrate some concepts that can easily be confused.

(1) Telescoping series: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.
Notice, this looks a lot like a partial fraction, so $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. So we get,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right) = 1-\frac{1}{n+1} \to 1 \quad \text{as} \quad n\to\infty.$$

(2) Harmonic series: $\sum_{n=1}^{\infty} 1/n$.
Notice, that the sequence $1/n$ converges to 0 as $n \to \infty$, however we will show that the <u>series</u> diverges. In order to do this we calculate the partial sums and put estimates on them, $s_1 = 1$, $s_2 = 1 + 1/2$, $s_4 = 1+1/2+(1/3+1/4) > 1+1/2+(1/4+1/4) = 2$, $s_8 > 1+3/2$, $s_{16} > 1+4/2$, $s_{32} > 1 + 5/2$, $s_{64} > 1 + 6/2$. So, $s_{2^n} > 1 + n/2 \Rightarrow \infty$ as $n \to \infty$. So, by definition, the series diverges.

The following theorems give us a frame work to prove divergence but <u>NOT</u> convergence.

Theorem 2. *If the series* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

Proof. We can calculate the partial sums,

$$s_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$$
$$s_{n-1} = a_1 + a_2 + \cdots + a_{n-2} + a_{n-1}$$

Now, if we subtract the two, we get $s_n - s_{n-1} = a_n$, so we have a representation of a_n from the partial fractions. Now, since the series converges, the partial sums converge to exactly that sum, so $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} s_{n-1} = s$. Therefore, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - s_{n-1} = s - s = 0$. □

Corollary 1. *If* $\lim_{n\to\infty} a_n \neq 0$ *or doesn't exist, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Ex: Show $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.
Solution: We can just show that the sequence a_n doesn't go to zero.

$$\lim_{n\to\infty} \frac{n^2}{5n^2+4} = \lim_{n\to\infty} \frac{1}{5+4/n^2} = \frac{1}{5}.$$

Here are some properties of sums that we should keep in mind,

Theorem 3. *If $\sum a_n$ and $\sum b_n$ converge, $\sum ca_n$ and $\sum a_n \pm b_n$ converge, and*

$$a)\ \sum ca_n = c\sum a_n \qquad \text{and} \qquad b)\ \sum(a_n \pm b_n) = \sum a_n \pm \sum b_n. \tag{3}$$

Ex: Does $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$ converge? If so, find the sum.
Solution: First we find the two sums individually,

$$\sum_{n=1}^{\infty} \frac{3}{n(n+1)} = 3\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = 3.$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} = \frac{1/2}{1-1/2} = 1.$$

So, the series converges to $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right) = 4.$

10.3 Integral test

I can't quite give the full motivation as I did in class, but basically if we look at an integral we can approximate it by a sum. Since integrals and sums are so intimately connected, we can make a conclusion about the sum from evaluating the integral.

Ex: Lets look at $\sum_{n=1}^{\infty} 1/n^2$. If we look at the partial sums we have $\lim_{n\to\infty} s_n < 1 + \int_1^{\infty} \mathrm{d}x/x^2$ because the partial sums will just be right Riemann sums after the first one. So, if the integral converges the series will also converge. But we already know the integral converges since $p > 1$. So, the series too converges.

We have a similar result for series that diverge, but let's not go over that and get straight to the test. To see for yourself test it out with $\sum_{n=1}^{\infty} 1/n$.

Theorem 4. *Integral test: Suppose f is continuous, positive, and decreasing on $[1,\infty)$ and let $a_n = f(n)$, then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the integral $\int_1^{\infty} f(x)dx$ also converges, i.e.*

$$\int_1^{\infty} f(x)dx \text{ converges } \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges.} \tag{4}$$

$$\int_1^{\infty} f(x)dx \text{ diverges } \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges.} \tag{5}$$

(1) Test $\sum_{n=1}^{\infty} 1/(n^2+1)$
Solution: We integrate:

$$\int_1^{\infty} \frac{\mathrm{d}x}{x^2+1} = \lim_{t\to\infty} \int_1^{t} \frac{\mathrm{d}x}{x^2+1} = \lim_{t\to\infty} \tan^{-1} x|_1^t = \lim_{t\to\infty} (\tan^{-1} t - \pi/4) = \frac{\pi}{4}.$$

(2) For what values of p does $\sum_{n=1}^{\infty} 1/n^p$ converge?
Solution: We have to integrate $\int_1^{\infty} \mathrm{d}x/x^p$, but we already know this converges for $p > 1$, and by the integral test, the series too converges for $p > 1$ and diverges otherwise. This is called a p-series.

Theorem 5. *P-series: The series $\sum_{n=1}^{\infty} 1/n^p$ converges for $p > 1$, and diverges otherwise.*

(1) $\sum_{n=1}^{\infty} 1/n^3$ converges because $p = 3 > 1$.
(2) $\sum_{n=1}^{\infty} 1/n^{1/3}$ diverges because $p = 1/3 < 1$.
(3) Test $\sum_{n=1}^{\infty} (\ln n)/n$.
Solution: We have to integrate this,

$$\int_1^{\infty} \frac{\ln x}{x} \mathrm{d}x = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x} \mathrm{d}x = \lim_{t\to\infty} \frac{1}{2}(\ln x)^2 \Big|_1^t = \lim_{t\to\infty} \frac{1}{2}(\ln t)^2 = \infty$$

There will be times when we wont be able to find the sum of certain convergent series. In these cases it is beneficial to estimate the sum. Notice the bigger partial sum we take, the better the estimate, but how can we tell how good the estimate is? Since it converges, we can use two integrals to do this. Notice that $s \leq s_n + \int_n^{\infty} f(x)\mathrm{d}x$ and $s \geq s_n + \int_{n+1}^{\infty} f(x)\mathrm{d}x$ because these are like left and right hand Riemann estimates for integrals of monotonic functions.

Definition 2. Suppose $\sum_{n=1}^{\infty} a_n = s$, and s_n are it's partial sums. Then the remainder of the n^{th} partial sum is $R_n = s - s_n$.

Theorem 6. *Remainder: Consider $\sum^{\infty} a_n = s$. Suppose $f(x) = a_k$, where f is continuous, positive, and decreasing for $x \geq n$, then*

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_n^{\infty} f(x)\,dx. \tag{6}$$

Ex: Consider $\sum_{n=1}^{\infty} 1/n^3$.
(a) Find the maximum error for $n = 10$.
Solution: We just plug this into the formula,

$$R_{10} \leq \int_{10}^{\infty} \frac{\mathrm{d}x}{x^3} = \lim_{t\to\infty} \int_{10}^{t} \frac{\mathrm{d}x}{x^3} = \lim_{t\to\infty} \frac{-1}{2x^2}\Big|_{10}^{t} = \lim_{t\to\infty} \frac{-1}{2t^2} - \frac{-1}{2*(10)^2} = \frac{1}{200} = .005.$$

(b) How many terms must we take for $R_n \leq .0005$?
Solution: Here we bound our formula and see what n has to be,

$$R_n \leq \int_n^{\infty} \frac{\mathrm{d}x}{x^3} = \frac{1}{2n^2} < .0005 \Rightarrow n^2 > \frac{1}{.001} = 1000 \Rightarrow n > \sqrt{1000} \approx 31.6.$$

So, we must take 32 terms.
(c) Now, notice if we add s_n to both sides of the inequality we get bounds on the exact solution, i.e. $s_10 \approx 1.1975$, so for $n = 10$.

$$\begin{aligned} s_10 + \int_{11}^{\infty} f(x)\mathrm{d}x &\leq R_10 + s_10 \leq s_10 + \int_{10}^{\infty} f(x)\mathrm{d}x \\ &\Rightarrow 1.1975 + \frac{1}{242} \leq s \leq 1.1975 + \frac{1}{200} \\ &\Rightarrow 1.2016 \leq \sum_{n=1}^{\infty} 1/n^3 \leq 1.2025. \end{aligned}$$