10.4 Comparison tests

This is very similar to integral comparison tests.

Theorem 1. *Direct Comparison: Suppose* $\sum a_n$ *and* $\sum b_n$ *have positive terms, then*

(i) *If* $\sum b_n$ *converges and* $a_n \leq b_n$ *for all* n*, then* $\sum a_n$ *also converges.*
(ii) *If* $\sum b_n$ *diverges and* $a_n \geq b_n$ *for all* n*, then* $\sum a_n$ *also diverges.*

State whether the following converge or diverge, and state the reasoning.

(1) $\sum_{n=1}^{\infty} 1/(2^n + 1)$.
Solution: We know $\frac{1}{2^n+1} < \frac{1}{2^n}$. Therefore, since $\sum_{n=1}^{\infty} 1/2^n$ converges because of p-series, where $p > 1$, $\sum_{n=1}^{\infty} 1/(2^n + 1)$ also converges by the direct comparison test.

(2) $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$.
Solution: Here as usual we take the highest power of the top and bottom. This will give us $5/2n^2$. We know the sum of this converges, so for direct comparison we would attempt to show that this is greater than our original sequence. This is easy to show since all the terms in the denominator are additive, so $\frac{5}{2n^2+4n+3} \leq \frac{5}{2n^2}$. Since $\sum_{n=1}^{\infty} 1/n^2$ converges by p-series because $p > 1$, the original series also converges by the direct comparison test.

(3) $\sum_{n=1}^{\infty} (\ln n)/n$.
Solution: Since $\ln n > 1$ for $n \geq 3$, $\frac{\ln n}{n} \geq \frac{1}{n}$ for $n \geq 3$. Further, since $\sum_{n=1}^{\infty} 1/n$ diverges by p-series because $p = 1$, by the direct comparison test, the original series converges as well. Notice that we only care about the tail end.

Notice that we can't use this test on something like $\sum_{n=1}^{\infty} 1/(2^n - 1)$, so we need the limit comparison test,

Theorem 2. *Limit comparison: Suppose* $\sum a_n$ *and* $\sum b_n$ *have positive terms, and* $\lim_{n\to\infty} a_n/b_n = c > 0$*, where* c *is a finite number. Then, either both* $\sum a_n$ *and* $\sum b_n$ *converge or both diverge. Further, if* $c = 0$ *and* $\sum b_n$ *converges, then* $\sum a_n$ *converges, and if* $c = \infty$ *and* $\sum b_n$ *diverges, then* $\sum a_n$ *diverges.*

State whether the following converge or diverge, and state the reasoning.

(1) $\sum_{n=1}^{\infty} 1/(2^n - 1)$.
Solution: Again we take the highest power of both the top and the bottom, i.e. $1/2^n$. Taking the limit gives,

$$\lim_{n\to\infty} \frac{1/(2^n-1)}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{1}{1-1/2^n} = 1 > 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges by geometric series because $|1/2| < 1$, by the limit comparison test $\sum_{n=1}^{\infty} 1/(2^n - 1)$ also converges.

(2) $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$.
Solution: As before we take the largest power of the numerator and largest part of the denominator, i.e. $2n^2/n^{5/2} = 2/\sqrt{n}$. Taking the limit gives,

$$\lim_{n\to\infty} \frac{(2n^2+3n)/\sqrt{5+n^5}}{2/\sqrt{n}} = \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{2\sqrt{5+n^5}} = \lim_{n\to\infty} \frac{2+3/n}{2\sqrt{5/n^5+1}} = 1.$$

Since $\sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges by p-series because $p<1$, by the limit comparison test, $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ also diverges.

10.5 Ratio and Root Tests

Sometimes we need to bring out the big guns,

Theorem 3. *Ratio test: Consider* $\sum a_n$*, and suppose* $\lim_{n\to\infty} |a_{n+1}/a_n| = L$*, then*

a) *If* $L<1$*, then* $\sum a_n$ *converges absolutely,*
b) *If* $L>1$*, then* $\sum a_n$ *diverges,*
c) *and if* $L=1$*, the test is inconclusive.*

State whether the following converge or diverge, and state the reasoning.

(1) $\sum_{n=1}^{\infty} n^3/3^n$.
Solution: We apply the ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}\right| = \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1+\frac{1}{n}\right)^3.$$

Taking the limit of this gives, $\lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^3 = \frac{1}{3} < 1$. Therefore, by the ratio test, $\sum_{n=1}^{\infty} n^3/3^n$ converges absolutely.

(2) $\sum_{n=1}^{\infty} n^n/n!$.
Solution: We apply the ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}\right| = \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n.$$

Taking the limit gives,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \exp\left[\lim_{n\to\infty} n\ln(1+\frac{1}{n})\right],$$

Now, we look at just the inside,

$$\lim_{n\to\infty} n\ln(1+\frac{1}{n}) = \lim_{n\to\infty} \frac{\ln(1+\frac{1}{n})}{1/n} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1.$$

then,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1.$$

Therefore, by the ratio test, $\sum_{n=1}^{\infty} n^n/n!$ diverges.

Theorem 4. *Root test: Consider* $\sum a_n$*, and suppose* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$*, then*

a) *If* $L < 1$*, then* $\sum a_n$ *converges absolutely,*
b) *If* $L > 1$*, then* $\sum a_n$ *diverges,*
c) *and if* $L = 1$*, the test is inconclusive.*

State whether the following converge or diverge, and state the reasoning.

(1) $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.
Solution: We apply the root test,

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left|\left(\frac{2n+3}{3n+2}\right)^n\right|} = \frac{2n+3}{3n+2} = \frac{2+3/n}{3+2/n}.$$

Taking the limit gives, $\lim_{n\to\infty} \frac{2+3/n}{3+2/n} = \frac{2}{3} < 1$. Therefore, by the root test, $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ converges absolutely.

10.6 Alternating Series and Absolute Convergence

A series that has alternating signs, i.e. $\sum(-1)^n b_n$, which means we get cancellations. We, of course, have a test for these situations.

Theorem 5. *Alternating series test: Consider the series* $\sum(-1)^n b_n$*, where* $b_n > 0$*, and*

(i) $b_{n+1} \le b_n$ *(i.e.* b_n *are decreasing) for all* $n > N$*, where* $N \in \mathbb{N}$
(ii) $\lim_{n\to\infty} b_n = 0$ *(i.e. the series* b_n *converges to* 0*),*

then $\sum(-1)^n b_n$ *converges.*

State whether the following converge or diverge, and state the reasoning.

(1) Alternating harmonic series: $\sum_{n=1}^{\infty}(-1)^{n-1}/n$.
Solution: First we take the limit, $\lim_{n\to\infty} 1/n = 0$. Now, we show the $(n+1)^{\text{th}}$ term is smaller than the n^{th} term, $1/(n+1) \le 1/n$ for all n. Therefore, by the alternating series test, $\sum_{n=1}^{\infty}(-1)^{n-1}/n$ converges.
(2) $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$.
Solution: Lets take the limit of the sequence, $\lim_{n\to\infty} \frac{3n}{4n-1} = \lim_{n\to\infty} \frac{3}{4-1/n} = \frac{3}{4}$. Since this does not converge to 0, the series will diverge.
(3) $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n^2}{n^3+1}$.
Solution: Taking the limit gives, $\lim_{n\to\infty} \frac{n^2}{n^3+1} = 0$.

Error estimation:
This may or may not show up on the exam. If it does show up it will be a minor question, so know how to do this, but don't put too much effort into it.

If $\sum(-1)^n b_n$ satisfies the alternating series test, then the remainder $|R_n| \le b_{n+1}$.

Ex: Approximate the sum of $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ to three decimal places.
Brief Solution: We see that $b_7 = 1/5040 < 1/5000 < .0002$, so s_6 (the sixth partial sum) is correct up to three decimal places, which is $s \approx s_6 = .368$.

Definition 1. The series $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges. Otherwise, if $\sum a_n$, but $\sum |a_n|$ diverges, then $\sum a_n$ is said to be conditionally convergent.

State whether the following are absolutely convergent, conditionally convergent, or divergent.

(1) $\sum_{n=1}^{\infty}(-1)^{n-1}/n^2$.
Taking the absolute value gives, $\sum_{n=1}^{\infty}|(-1)^{n-1}/n^2| = \sum_{n=1}^{\infty} 1/n^2$. We know this converges by p-series because $p > 1$. Therefore, $\sum_{n=1}^{\infty}(-1)^{n-1}/n^2$ is absolutely convergent.

(2) $\sum_{n=1}^{\infty}(-1)^{n-1}/n$.
Taking the absolute value gives, $\sum_{n=1}^{\infty}|(-1)^{n-1}/n| = \sum_{n=1}^{\infty} 1/n$. We know this diverges by p-series because $p = 1$. However, $\sum_{n=1}^{\infty}(-1)^{n-1}/n$ converges by the alternating series test, which we showed further up. Therefore, $\sum_{n=1}^{\infty}(-1)^{n-1}/n$ converges conditionally.

Theorem 6. *If $\sum a_n$ converges absolutely, then it converges.*

Ex: Does $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converge?

Solution: Lets look at the sum of the absolute values, $\sum_{n=1}^{\infty}\left|\frac{\cos n}{n^2}\right|$. Now, $\left|\frac{\cos n}{n^2}\right| \leq \frac{1}{n^2}$. We know that $\sum_{n=1}^{\infty} 1/n^2$ converges by p-series because $p > 1$. Hence, $\sum_{n=1}^{\infty}\left|\frac{\cos n}{n^2}\right|$ also converges. Therefore since $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converges absolutely, it converges.