FALL 2015 SOLUTIONS

(1) (a) We need to change this to the following series,

$$\frac{4}{3}\sum_{n=0}^{\infty}\frac{4^n}{(n+2)^n}.$$

Then we use Root test,

$$\lim_{n\to\infty}\sqrt[n]{\frac{4^n}{(n+2)^n}}=\lim_{n\to\infty}\frac{4}{n+2}=0<1$$

Therefore, by root test the sum converges.

(b) Here we can use direct comparison,

$$\frac{\cos^2 n+1}{n^2+1}\leq\frac{2}{n^2}.$$

and $\sum_{n=1}^{\infty}2/n^2$ converges since $p>1$, therefore by DCT the series converges.

(2) (a) We use LCT with π/n,

$$\lim_{n\to\infty}\frac{\sin(\pi/n)}{\pi/n}=1.$$

and $\sum_{n=1}^{\infty}\pi/n$ diverges since $p\leq 1$, therefore by LCT the series diverges.

(b) We use LCT with $1/2^n$,

$$\lim_{n\to\infty}\frac{(n+2^{-n})\,/\,(n2^n-1)}{1/2^n}=\lim_{n\to\infty}\frac{n2^n+1}{n2^n-1}=1.$$

and $\sum_{n=1}^{\infty}(1/2)^n$ converges since $1/2<1$, hence the series converges by LCT.

(3) (a) Here we use Ratio test,

$$\lim_{n\to\infty}\left|\frac{\ln(n+1)(3/4)^{n+1}}{\ln(n)(3/4)^n}\right.=\lim_{n\to\infty}\frac{3}{4}\frac{1/(n+1)}{1/n}=\lim_{n\to\infty}\frac{3}{4}\frac{n}{n+1}=\frac{3}{4}<1.$$

Therefore, by the ratio test the series converges absolutely.

(b) Here we just take the limit,

$$\lim_{n\to\infty}\frac{1}{1+e^{1/n}}=\frac{1}{2}\neq 0$$

Therefore, it diverges.

(4) Lets first try absolute convergence using LCT with $1/\sqrt{n}$,

$$\lim_{n\to\infty}\frac{n/\sqrt{n^3+3}}{1/\sqrt{n}}=\lim_{n\to\infty}\frac{n^{3/2}}{\sqrt{n^3+3}}=1.$$

and $\sum_{n=1}^{\infty}1/\sqrt{n}$ diverges since $p\leq 1$. Now, lets do AST, $\lim_{n\to\infty}n/\sqrt{n^3+3}=0$, so we proceed with showing decreasing,

$$\left(\frac{n}{\sqrt{n^3+3}}\right)'=\frac{\sqrt{n^3+3}-\frac{1}{2}(n^3+3)^{-1/2}3n^3}{n^3+3}=\frac{n^3+3-\frac{3n^3}{2}}{(n^3+3)^{3/2}}.$$

Notice, $3-n^3/2<0$ for all $n\geq 1$, so the series is conditionally convergent by AST.

(5) As usual we do ratio test,

$$\lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{n!}\cdot\frac{(n-1)!}{(x-1)^n}\right| = \lim_{n\to\infty}\frac{1}{n}|x-1| = 0$$

This shows that $R=\infty$ and $x\in(-\infty,\infty)$.

(6) Again we do ratio test,

$$\lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{(n+3)3^{n+1}}\cdot\frac{n3^n}{(x+2)^n} = \lim_{n\to\infty}\frac{n}{n+1}\right|\frac{x+2}{3}\left| = \left|\frac{x+2}{3}\right| < 1 \Rightarrow |x+2|<3.$$

This shows, $R=3$, and $x\in(-5,1)$. Now let's test the end points. For $x=5$, $\sum_{n=1}^{\infty}1/n$ diverges since $p\le 1$. For $x=1$, $\sum_{n=1}^{\infty}(-1)^n/n$ converges conditionally by AST, since $\lim_{n\to\infty}1/n=0$ and $1/n>1/(n+1)$. Therefore, $x\in(-5,1]$.

(7) The Taylor series of $\cos x$ about $x=0$ is,

$$\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \Rightarrow \cos 2x = \sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n}}{(2n)!} = 1-2x^2+\frac{2}{3}x^4+\cdots$$

Therefore,

$$(1+x^2)\cos(2x) \approx (1+x^2)\left(1-2x^2+\frac{2}{3}x^4\right) = 1-x^2-\frac{4}{3}x^4.$$

(8) Here we just take derivatives and compute, with $f(\pi/4)=1$.

$$f'(x) = 2\cos(2x)\Big|_{x=\pi/4} = 0, \qquad f''(x) = -4\sin(2x)\Big|_{x=\pi/4} = -4$$

$$f'''(x) = -8\cos(2x)\Big|_{x=\pi/4} = 0, \qquad f^{(4)}(x) = 16\sin(2x)\Big|_{x=\pi/4} = 16$$

$$\Rightarrow f(x) \approx 1-2(x-\pi/4)^2+\frac{2}{3}(x-\pi/4)^4.$$

(9) (a) Again we take derivatives and compute, with $f(1)=e$.

$$f'(x) = e^x + xe^x\Big|_{x=1} = 2e,\; f''(x) = 2e^x + xe^x\Big|_{x=1} = 3e$$

$$\Rightarrow f(x) \approx e + 2e(x-1) + \frac{3}{2}e(x-1)^2.$$

(b) Here we take the third derivative and evaluate it at $x=2$ and evaluate $|x-1|^3$ at $x=2$,

$$f'''(x) = 3e^x + xe^x\Big|_{x=2} = 5e^2 \Rightarrow |R_2| = \frac{5e^2}{6}$$