11.3 and 11.4 Polar Coordinates and Sketching

For polar coordinates r is the distance from the origin and θ is the angle from the x-axis. The ordered pairs are denoted (r, θ) and $x = r\cos\theta$, $y = r\sin\theta$, which means $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$.

Ex: We plotted the following points in class:
a) $(1, 5\pi/4)$ b) $(2, 3\pi)$ c) $(2, -2\pi/3)$ d) $(-3, 3\pi/4)$.

Ex: Convert $(2, \pi/3)$ from Polar to Cartesian coordinates. Plugging these quantities into the formulas gives $(1, \sqrt{3})$.

Ex: Convert $(1, -1)$ from Cartesian to Polar coordinates. Plugging these quantities into the formulas gives $(\sqrt{2}, -\pi/4)$.

The main thing we will do with Polar coordinates is sketching and analyzing polar functions. A polar function is a function of the form $r = f(\theta)$ i.e. how the radius changes as a function of the angle.

We sketched $r = 2\cos\theta$, $r = 1 + \sin\theta$, and $r = \cos 2\theta$ in class. Refer to the link that I sent. I'll try and remember to send the link again with this email.

Ex: Consider $r = 1 + \sin\theta$, $0 \le \theta \le 2\pi$.
(a) Find the slope of the tangent line at $\theta = \pi/3$,
(b) Find the points where the tangent lines are horizontal/vertical.
(a) **Solution**: We must compute $\mathrm{d}y/\mathrm{d}x$ in the usual manner from last section,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta} = \frac{\cos\theta(1+2\sin\theta)}{1-2\sin^2\theta-\sin\theta} = \frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}.$$

Plugging in $\theta = \pi/3$ gives $\mathrm{d}y/\mathrm{d}x = -1$.

(b) **Solution**: We compute the respective derivatives and equate them to zero,

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = \cos\theta(1+2\sin\theta) = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}.$$

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = (1+\sin\theta)(1-2\sin\theta) = 0 \Rightarrow \theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}.$$

We can make conclusions about all the points except for the duplicate. For the duplicate we must take the limit

$$\lim_{\theta\to\frac{3\pi}{2}^-} \frac{\mathrm{d}y}{\mathrm{d}x} = -\lim_{\theta\to\frac{3\pi}{2}^+} \frac{\mathrm{d}y}{\mathrm{d}x} = \infty$$

So, the horizontal points correspond to $\theta = \pi/2,\ 7\pi/6,\ 11\pi/6$ and the vertical points correspond to $\theta = \pi/6,\ 5\pi/6,\ 3\pi/2$.

11.5 Areas and Lengths in Polar coordinates

We know the area of a sector is $A = r^2\theta/2$, so if $r = f(\theta)$ the area of a wedge in a polar curve will be,

$$\mathrm{A} = \int_\alpha^\beta \frac{1}{2} r^2 \mathrm{d}\theta = \int_\alpha^\beta \frac{1}{2} f(\theta)^2 \mathrm{d}\theta. \tag{1}$$

We can derive the arc length in the usual manner, but it does get very tedious, so one should go straight to the formula,

$$\text{Ł} = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta. \tag{2}$$

(1) Find the area enclosed by $r = \cos 2\theta$ and the x-axis.
Solution: We plug this straight into the formula,

$$\mathrm{A} = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta \mathrm{d}\theta = \int_0^{\pi/4} \cos^2 2\theta \mathrm{d}\theta = \int_0^{\pi/4} \frac{1}{2}(1+\cos 4\theta)\mathrm{d}\theta = \frac{1}{2}\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/4} = \frac{\pi}{8}.$$

(2) Find the area inside $r = 3\sin\theta$ and outside $r = 1+\sin\theta$.
Solution: The first thing we have to do is find where they intersect so that we can figure out our limits,

$$3\sin\theta = 1 + \sin\theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}.$$

Then, we plug into our formula,

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6}[(3\sin\theta)^2 - (1+\sin\theta)^2]\mathrm{d}\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6}(8\sin^2\theta - 1 - 2\sin\theta)\mathrm{d}\theta$$
$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6}(3 - 4\cos 2\theta - 2\sin\theta)\mathrm{d}\theta = \frac{1}{2}[\theta - 2\sin 2\theta + 2\cos\theta]_{\pi/6}^{5\pi/6} = \pi.$$

(3) Find the length of $r = 1 + \sin\theta$ for $0 \leq \theta \leq 2\pi$.
Solution: We plug in to our formula to get,

$$Ł = \int_0^{2\pi}\sqrt{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta}\mathrm{d}\theta = \int_0^{2\pi}\sqrt{2 + 2\sin\theta}\frac{\sqrt{2 - 2\sin\theta}}{2 - 2\sin\theta}\mathrm{d}\theta$$
$$= \int_0^{2\pi}\frac{\sqrt{4 - 4\sin^2\theta}}{\sqrt{2 - 2\sin\theta}}\mathrm{d}\theta = \int_0^{2\pi}\frac{2\cos\theta\mathrm{d}\theta}{\sqrt{2 - 2\sin\theta}}.$$

Now this is an improper integral at $\theta = \pi/2$, so we would need to take limits. Assuming we do this our final answer should be 8.