6.1 Volumes using Cross-sections

The volume of any solid with cross-sectional area $A(x)$ from $x = a$ to $x = b$ is

$$V = \int_a^b A(x)\mathrm{d}x \tag{1}$$

Disks.

Specifically, for rotations using disks (i.e. regions without gaps) about the x and y axes are respectively,

$$V = \int_a^b \pi R(x)^2 \mathrm{d}x \tag{2}$$

$$V = \int_a^b \pi R(y)^2 \mathrm{d}y \tag{3}$$

(1) Find the volume of a sphere centered at the origin with radius r.
Solution: Consider circular cross-sections, i.e. circles cutting the sphere perpendicular to the x-axis of radius y. Then, by the distance formula (or Pythagorean theorem), $R(x)^2 = y^2 = r^2 - x^2$. So,

$$V = \int_a^b \pi R(x)^2 \mathrm{d}x = \int_{-r}^r \pi(r^2 - x^2)\mathrm{d}x = \frac{4}{3}\pi r^3.$$

(2) Find the volume of a region bounded by $y = \sqrt{x}$, $x = 0$, and $x = 1$ about the x-axis.

$$\textbf{Solution}: R(x) = \sqrt{x} \Rightarrow V = \int_0^1 \pi x \mathrm{d}x = \frac{\pi}{2}$$

(3) Find the volume of a region bounded by $y = x^3$, $x = 0$, and $y = 8$ about the y-axis.
Solution: Since we are revolving around the y-axis we need to solve for x as a function of y, i.e. $x = y^{1/3}$, then

$$R(y) = y^{1/3} \Rightarrow V = \int_0^8 \pi y^{2/3} \mathrm{d}y = \frac{96}{5}\pi$$

Washers.

For rotations using washers (i.e regions with gaps) about the x and y axes are respectively,

$$V = \int_a^b \pi[R(x)^2 - r(x)^2]\mathrm{d}x \tag{4}$$

$$V = \int_a^b \pi[R(y)^2 - r(y)^2]\mathrm{d}y \tag{5}$$

Where R is the radius of the larger region, and r is the radius of the smaller region (i.e. gap).

(1) Find the volume of the region between $y = x$ and $y = x^2$ revolved about the x-axis.
Solution: Since $y = x$ is on top, it sweeps out a larger region than $y = x^2$, so $R(x) = x$ and $r(x) = x^2$. Then,

$$V = \int_a^b \pi[R(x)^2 - r(x)^2]\mathrm{d}x = \int_0^1 \pi[x^2 - x^4]\mathrm{d}x = \frac{2\pi}{15}$$

(2) What if we revolve it around $y = 2$?
Solution: Since we are revolving about an axis above our region, the function on the bottom will now sweep out a larger region. This can be seen by sketching the curves. So, $R(x) = 2 - x^2$ and $r(x) = 2 - x$. Then,

$$V = \int_a^b \pi[R(x)^2 - r(x)^2]\mathrm{d}x = \int_0^1 \pi[(2 - x^2)^2 - (2 - x)^2]\mathrm{d}x = \frac{8\pi}{15}$$

(3) Now lets revolve it around $x = -1$.
Solution: First we must solve for x as a function of y now. We get, $x = y$ and $x = \sqrt{x}$, respectively. Since we are revolving about an axis to the left of our region, the function on the right will sweep out a larger region. So, $R(y) = \sqrt{y} - (-1)$ and $r(y) = y - (-1)$. Then,

$$V = \int_a^b \pi[R(y)^2 - r(y)^2]\mathrm{d}y = \int_0^1 \pi[(1 + \sqrt{y})^2 - (1 + y)^2]\mathrm{d}y = \frac{\pi}{2}$$

Tougher examples.

(1) Find the volume of a region such that the base is a circle centered at the origin of radius $r = 1$ and cross-sections perpendicular to the x-axis that are equilateral triangles.
Solution: Notice the base of the triangle will be $2y$, and since they are equilateral triangles we can split the triangle in half to get a "30, 60, 90" triangle with base y, so the height of the triangles will be $\sqrt{3}y$. This gives us a cross-sectional area of $A(y) = \sqrt{3}y^2$, but we have cross-sections perpendicular to the x-axis, so we need $A(x)$. Notice a circle is given by the equation $y^2 + x^2 = r^2$, but $r = 1$, so $y^2 = 1 - x^2$. Hence, $A(x) = \sqrt{3}(1 - x^2)$. Then,

$$V = \int_{-1}^{1} \sqrt{3}(1 - x^2)\mathrm{d}x = \frac{4\sqrt{3}}{3}.$$

(2) Find the volume of a square pyramid such that the sides of the square are length L and height h.
Solution: There are many ways to do this, but perhaps the easiest is to consider a triangle with the head at the origin and base at $x = L$ with square cross-sections perpendicular to the x-axis. So, at an arbitrary x, the cross-section is a square of length, say ℓ. Notice, the entire triangle and a triangle at any arbitrary x will be similar triangles, and hence have the same ratios. Therefore, $\frac{x}{h} = \frac{\ell}{L}$, i.e. the ratio of the heights equal the ratio of the lengths. This gives $\ell = \frac{L}{h}x$. Then, $A(x) = (Lx/h)^2$, and

$$V = \int_{0}^{h} \frac{L^2}{h^2}x^2\mathrm{d}x = \frac{1}{3}L^2h$$

(3) Find the volume of a region such that the base is a semicircle centered at the origin with radius $r = 4$ and cross-sections perpendicular to the x-axis that are "30, 60, 90" triangles - 30° with respect to the x-axis.
Solution: Notice the base at any arbitrary x will be of length y. The height of the triangle will be $y/\sqrt{3}$, then $A(y) = \frac{1}{2\sqrt{3}}y^2$. Since $r = 4$, $y^2 = 16 - x^2$. Then, $A(x) = \frac{1}{2\sqrt{3}}(16 - x^2)$. Then,

$$V = \int_{-4}^{4} \frac{1}{2\sqrt{3}}(16 - x^2)\mathrm{d}x = \frac{128}{3\sqrt{3}}.$$

6.2 Cylindrical Shells

Another method to do volumes of revolutions is through cylindrical shells. This method is a lot less intuitive, and hence requires more practice. I wonder if any of you guys are actually reading this... Basically think of infinitesimal cylinders filling up a region. We know the area of the side of the cylinder is $A = 2\pi rh$. So, by summing up these infinitesimal cylinders we get the following formulas for rotation about the y-axis and x-axis respectively,

$$V = \int_a^b 2\pi x h(x) \mathrm{d}x \tag{6}$$

$$V = \int_a^b 2\pi y h(y) \mathrm{d}y \tag{7}$$

(1) Find the volume of the region bounded by $y = 2x^2 - x^3$ and $y = 0$ revolved about the y-axis.
Solution: Here the radius of each cylinder will be $r = x$ and the height will be $h = y = 2x^2 - x^3$. Then,

$$V = \int_a^b 2\pi x f(x) \mathrm{d}x = \int_0^2 2\pi x(2x^2 - x^3) \mathrm{d}x = \frac{16}{5}\pi$$

(2) Find the volume of the region bounded by $y = x$ and $y = x^2$ revolved about the y-axis.
Solution: The radius is $r = x$ and the "height" is $h = x - x^2$, then

$$V = \int_a^b 2\pi x f(x) \mathrm{d}x = \int_0^1 2\pi x(x - x^2) \mathrm{d}x = \frac{\pi}{6}$$

(3) Find the volume of the region bounded by $y = \sqrt{x}$, $x = 0$, and $x = 1$ revolved about the x-axis.
Solution: The radius is $r = y$ and the "height" is $h = 1 - y^2$, then

$$V = \int_a^b 2\pi y f(y) \mathrm{d}y = \int_0^1 2\pi y(1 - y^2) \mathrm{d}y = \frac{\pi}{2}$$

(4) Find the volume of a region bounded by $y = x - x^2$ and $y = 0$ revolved about the line $x = 2$.
Solution: Since our axis of revolution is towards the right, the radius of our cylinders will be $r = 2 - x$ and the height is $h = y = x - x^2$. Hence, our area is $A(x) = 2\pi(2 - x)(x - x^2)$. Then,

$$V = \int_0^1 2\pi(2 - x)(x - x^2) \mathrm{d}x = \frac{\pi}{2}$$

6.3 Arc Length

Arc length is just the sum of infinitesimally small pieces of an arc, so we can derive the formula:

$$L = \int_a^b \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}. \tag{8}$$

This can be parametrized in many ways, but there are two main ways:

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, \mathrm{d}x \quad \text{if } f \in C^1([a,b]), \tag{9}$$

$$L = \int_c^d \sqrt{1 + g'(y)^2}\, \mathrm{d}y \quad \text{if } g \in C^1([c,d]). \tag{10}$$

This means that we use the first formula if $y = f(x)$ has a continuous derivative on $[a,b]$ (the interval between which we are calculating arc length), and we use the second formula if $x = g(y)$ has a continuous derivative on $[c,d]$. If it has a continuous derivative for both, we may use either formula.

(1) Find the arc length of $y^2 = x^3$ between $(1,1)$ and $(4,8)$.
Solution: We simply differentiate and plug into the formula. The derivative is $f'(x) = \frac{3}{2}x^{1/2}$, then

$$L = \int_1^4 \sqrt{1 + \frac{9}{4}x}\mathrm{d}x$$

We solve this via "u-sub", where $u = 1 + \frac{9}{4}x$, then

$$L = \frac{4}{9}\int_{13/4}^{10} \sqrt{u}\mathrm{d}u = \frac{1}{27}(80\sqrt{10} - 13\sqrt{13}).$$

(2) Find the arc length of $y^3 = x^2$ between $(0,0)$ and $(1,1)$.
Solution: We differentiate to get $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{3}x^{-1/3}$. This is clearly not continuous at $x = 0$, so we need to find another way. We can differentiate with respect to y instead, so $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{3}{2}y^{1/2}$, then

$$L = \int_0^1 \sqrt{1 + \frac{9}{4}y}\mathrm{d}y = \frac{4}{9}\int_1^{13/4} \sqrt{u} = \frac{1}{27}(13\sqrt{13} - 8).$$