8.4 Trigonometric Substitutions

Lets begin with an example,

Ex: Consider $\int \sqrt{1-x^2}\mathrm{d}x$.

Solution: This reminds us of the identity $1-\sin^2\theta = \cos^2\theta$, so lets use the substitution $x=\sin\theta \Rightarrow \mathrm{d}x = \cos\theta\mathrm{d}\theta$.

$$\int \sqrt{1-x^2}\mathrm{d}x = \int \sqrt{1-\sin^2\theta}\cos\theta\mathrm{d}\theta = \int \sqrt{\cos^2\theta}\cos\theta\mathrm{d}\theta = \int \cos^2\theta\mathrm{d}\theta$$
$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C = \frac{\theta}{2} + \sin\theta\cos\theta = \frac{1}{2}\sin^{-1}x + x\sqrt{1-x^2}.$$

The tricky part is going from θ to x. We see that since $x=\sin\theta$, $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$. We can also use our right triangles to help us.

Let us employ a table of trig substitutions that will help us with the decision making process.

Expression	Substitution	Identity
$\sqrt{a^2-x^2}$	$x=a\sin\theta$, $-\pi/2\le\theta\le\pi/2$	$1-\sin^2\theta=\cos^2\theta$
$\sqrt{a^2+x^2}$	$x=a\tan\theta$, $-\pi/2<\theta<\pi/2$	$1+\tan^2\theta=\sec^2\theta$
$\sqrt{x^2-a^2}$	$x=a\sec\theta$, $0\le\theta<\pi/2$, $\pi\le\theta<3\pi/2$	$\sec^2\theta-1=\tan^2\theta$

(1) $I=\int \frac{\sqrt{9-x^2}}{x^2}\mathrm{d}x$.

Solution: This is of the form of the first case, so we use the substitution: $x=3\sin\theta \Rightarrow \mathrm{d}x = 3\cos\theta\mathrm{d}\theta$,

$$I = \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}3\cos\theta\mathrm{d}\theta = \int \frac{9\cos^2\theta}{9\sin^2\theta}\mathrm{d}\theta = \int \cot^2\theta\mathrm{d}\theta = \int (\csc^2\theta - 1)\mathrm{d}\theta = -\cot\theta - \theta + C.$$

Now we must plug back in for θ. Since $x=3\sin\theta$, $\sin\theta = x/3$, and we recall that sine is opposite over hypotenuse and cotangent is adjacent over opposite. We may denote the opposite side as x and the hypotenuse side as 3, then by the Pythagorean theorem the adjacent side is $\sqrt{9-x^2}$. This gives us, $\cot\theta = \sqrt{9-x^2}/x$, then

$$I = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

Note that we have to substitute back in only for indefinite integrals. For definite integrals it's easier to just change the limits.

(2) $I = \int \frac{\mathrm{d}x}{x^2\sqrt{x^2+4}}$.

Solution: This is of the form of the second case, so we substitute $x = 2\tan\theta \Rightarrow \mathrm{d}x = 2\sec^2\theta\mathrm{d}\theta$,

$$I = \int \frac{2\sec^2\theta\mathrm{d}\theta}{4\tan^2\theta\sqrt{4\tan^2\theta+4}} = \frac{1}{4}\int\frac{\sec\theta\mathrm{d}\theta}{\tan^2\theta} = \frac{1}{4}\int\frac{\cos\theta\mathrm{d}\theta}{\sin^2\theta}.$$

We solve this via u-sub with $u = \sin\theta \Rightarrow \mathrm{d}u = \cos\theta\mathrm{d}\theta$.

$$I = \frac{1}{4}\int\frac{\mathrm{d}u}{u^2} = -\frac{1}{4u} + C = -\frac{1}{\sin\theta} + C.$$

Since $x = 2\tan\theta$, $\tan\theta = x/2$, so $\sin\theta = x/\sqrt{x^2+4}$, then

$$I = -\frac{\sqrt{x^2+4}}{4x} + C.$$

(3) $I = \int \frac{x\mathrm{d}x}{\sqrt{x^2+4}}$.

Solution: Thought we had to use a trig substitution didn't ya? NOPE! U-Sub! Let $u = x^2+4 \Rightarrow \mathrm{d}u = 2x\mathrm{d}x$.

$$I = \frac{1}{2}\int\frac{\mathrm{d}u}{\sqrt{u}} = \sqrt{u} + C = \sqrt{x^2+4} + C.$$

This shows us that if we take a few seconds to think about a problem we can find a much easier solution.

(4) $I = \int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}$.

Solution: This is of the form of the third case, so we substitute $x = a\sec\theta \Rightarrow \mathrm{d}x = a\sec\theta\tan\theta$.

$$I = \int\frac{a\sec\theta\tan\theta\mathrm{d}\theta}{\sqrt{a^2\sec^2\theta - a^2}} = \int\frac{a\sec\theta\tan\theta\mathrm{d}\theta}{a\tan\theta} = \int\sec\theta\mathrm{d}\theta = \ln|\sec\theta+\tan\theta|+C.$$

Since $x = a\sec\theta$, $\sec\theta = x/a$, so $\tan\theta = \sqrt{x^2-a^2}/a$, then

$$I = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C.$$

Notice, $a\cosh x$ is an equivalent answer, but the preferable method is the way it was done here.

(5) $\int_0^{3\sqrt{3}/2} \frac{x^3 \mathrm{d}x}{(4x^2+9)^{3/2}}$.

Solution: This is of the form of the second case, but here we have a coefficient in front of the x term. We can either pull the 4 out and then start our calculations or we can see what x has to be with the 4 there. It's much easier to come up with a substitution for x that produces a desired result than to pull the coefficient out. Notice that we need the coefficient in front of the $\tan^2$ term (after substitution of course) to be 9, so we have that $4x^2 = 9\tan^2\theta$, then $x = (3/2)\tan\theta \Rightarrow \mathrm{d}x = (3/2)\sec^2\theta \mathrm{d}\theta$.

$$I = \left(\frac{3}{2}\right)^4 \int_0^{\pi/3} \frac{\tan^3\theta \sec^2\theta \mathrm{d}\theta}{(9\tan^2\theta + 9)^{3/2}} = \left(\frac{3}{2}\right)^4 \int_0^{\pi/3} \frac{\tan^3\theta \sec^2\theta \mathrm{d}\theta}{3^3\sec^3\theta} = \frac{3}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta}\mathrm{d}\theta$$
$$= \frac{3}{16}\int_0^{\pi/3} \frac{\sin^3\theta}{\cos^2\theta}\mathrm{d}\theta = \frac{3}{16}\int_0^{\pi/3} \frac{1-\cos^2\theta}{\cos\theta}\sin\theta \mathrm{d}\theta$$

This is our usual trig integral where $u = \cos\theta \Rightarrow \mathrm{d}u = -\sin\theta\mathrm{d}\theta$,

$$I = -\frac{3}{16}\int_1^{1/2} \frac{1-u^2}{u^2}\mathrm{d}u = \frac{3}{16}\left[u + \frac{1}{u}\right]_1^{1/2} = \frac{3}{32}.$$

(6) $I = \int \frac{x\mathrm{d}x}{\sqrt{3-2x-x^2}}$.

Solution: This one is going to take a bit of ingenuity. Lets tinker with $3-2x-x^2 = 3-(x^2+2x)$. Notice we can get a perfect square if we add a 1 to x^2+2x, but if we add a 1 we must also "subtract" a 1, so $3-2x-x^2 = 3-(x^2+2x+1)+1 = 4-(x+1)^2$. Now, let $u = x+1 \Rightarrow \mathrm{d}u = \mathrm{d}x$, then

$$I = \int \frac{(u-1)\mathrm{d}u}{\sqrt{4-u^2}}.$$

This is precisely the form of the first case, so we substitute $u = 2\sin\theta \Rightarrow \mathrm{d}u = 2\cos\theta\mathrm{d}\theta$.

$$\int \frac{2\sin\theta - 1}{\sqrt{4-4\sin^2\theta}} 2\cos\theta\mathrm{d}\theta = \int \frac{2\sin\theta - 1}{2\cos\theta} 2\cos\theta\mathrm{d}\theta = -2\cos\theta - \theta + C$$

Since $u = 2\sin\theta$, $\sin\theta = u/2$, then $2\cos\theta = \sqrt{4-u^2}$, so

$$I = -\sqrt{4-u^2} - \sin^{-1}\left(\frac{u}{2}\right) + C = \sqrt{3-2x-x^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C.$$

8.5 Integration of Rational Functions by Partial Fractions

Lets use the following example as motivation:

Ex: Consider $I = \int \frac{x+5}{x^2+x-2}\mathrm{d}x$.

Solution: Notice we can easily factor the denominator into $x^2 + x - 2 = (x-1)(x+2)$. Then we know that this looks like the common denominator of the sum of two fractions. Lets consider $\frac{1}{x-1} + \frac{1}{x+2} = \frac{2x+1}{(x-1)(x+2)}$. This is clearly not what we want, but this gives us an indication of the form of the fractions, namely

$$\frac{x+5}{x^2+x-2} = \frac{A}{x-1} + \frac{B}{x+2} = \frac{B(x-1)+A(x+2)}{(x-1)(x+2)} = \frac{(A+B)x+(2A-B)}{x^2+x-2}.$$

where A and B are some constants. Our task now is to solve for A and B. We notice that we must equate the numerators, i.e. $x+5 = (A+B)x + (2A-B)$, so by matching the coefficients we get two equations: $A+B=1$ and $2A-B=5$. From the first equation we have $B = 1-A$. Then plugging B into the second equation gives $2A-1+A = 3A-1 = 5 \Rightarrow A = 2$. Then, $B = 1-A = 1-2 = -1$. Now, we can plug these back into the fraction and put them back in the integral,

$$I = \int \frac{2\mathrm{d}x}{x-1} - \int \frac{\mathrm{d}x}{x+2} = 2\ln|x-1| - \ln|x+2| + C.$$

We digress slightly to do an example that does not involve partial fractions but that involves long division - a skill that will be very important for many of these types of problems,

Ex: $I = \int \frac{x^3+x}{x-1}\mathrm{d}x$.

Solution: By long division we get,

$$\frac{x^3+x}{x-1} = x^2+x+2+\frac{2}{x-1}.$$

If you're having trouble with long division please come see me, asap! Then, putting this back into the integral gives,

$$\int \frac{x^3+x}{x-1} = \int \left(x^2+x+2+\frac{2}{x-1}\right)\mathrm{d}x = \frac{1}{3}x^3+\frac{1}{2}x^2+2x+2\ln|x-1|+C.$$

Whenever the highest power in the numerator is greater than or equal to the highest power in the denominator we must use long division. Once it's in a form we can use, we can go ahead and use partial fractions. We can split the types of problems we will come across on the exam into four cases detailed bellow.

From this point on we will consider integrals of the type:

$$(1) \qquad \int f(x)\mathrm{d}x; \qquad f(x) = \frac{P(x)}{Q(x)}, \text{ where } P \text{ and } Q \text{ are polynomials.}$$

Case 1.

Suppose Q is a product of distinct linear factors, i.e. $Q = (a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$. Then,

$$(2) \qquad \frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}.$$

(1) Convert $\frac{x^2+2x-1}{2x^3+3x^2-2x}$ into partial fractions.
Solution: First we factor out the denominator,

$$2x^3+3x^2-2x = x(2x-1)(x+2).$$

Then,

$$\begin{aligned}\frac{x^2+2x-1}{2x^3+3x^2-2x} &= \frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2} \\ &= \frac{A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)}{x(2x-1)(x+2)} \\ &= \frac{(2A+B+2C)x^2+(3A+2B-C)x-2A}{2x^3+3x^2-2x}.\end{aligned}$$

Now we equate the numerators to find our constants,

$$x^2+2x-1 = (2A+B+2C)x^2+(3A+2B-C)x-2A.$$

Matching the coefficients give us the following equations,

$$\begin{aligned}2A+B+2C &= 1 \\ 3A+2B-C &= 2 \\ 2A &= 1\end{aligned}$$

The easiest one to solve for is $\boldsymbol{A = 1/2}$. Plugging this into the first equation gives, $B + 2C = 0 \Rightarrow B = -2C$. Plugging this into the second equation gives, $3/2 - 5C = 2 \Rightarrow -5C = 1/2 \Rightarrow$ $\boldsymbol{C = -1/10} \Rightarrow \boldsymbol{B = 1/5}$.

(2) Convert $\frac{1}{x^2-a^2}$ into partial fractions.
Solution:

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a} = \frac{(A+B)x+(A-B)a}{x^2-a^2}.$$

Matching the coefficients gives us $A+B=0 \Rightarrow A=-B$ straight away. Then we plug this into $(A-B)a = 2Aa = 1 \Rightarrow \boldsymbol{A = 1/2a} \Rightarrow \boldsymbol{B = -1/2a}$.

Case 2.

Suppose Q is a product of linear factors, some of which are repeated. Then, the repeated factors are of this form

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(ax+b)^r} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_r}{(ax+b)^r}. \tag{3}$$

(1) Convert $\frac{x^3-x+1}{x^2(x-1)^3}$ into partial fractions.
Solution: For this problem we simply put it into partial fractions form without finding the constants. Notice that the denominator is already in factored form.

$$\frac{x^3-x+1}{x^2(x-1)^3} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{B_1}{x-1} + \frac{B_2}{(x-1)^2} + \frac{B_3}{(x-1)^3}.$$