10.1 SEQUENCES (CONTINUED)

Monotonic Sequences. What if we couldn't take the limit of a sequence, but we knew some things about the function and wanted to analyze the behavior. The following definitions and theorem will help us deal with this.

Definition 1. A sequence a_n is called nondecreasing(think increasing) if $a_n \leq a_{n+1}$ for all $n \geq 1$, i.e. $a_1 \leq a_2 \leq a_3 \leq \cdots$. It is called nonincreasing (think decreasing) if $a_n \geq a_{n+1}$ for all $n \geq 1$, i.e. $a_1 \geq a_2 \geq a_3 \geq \cdots$. These types of sequences are collectively called monotonic sequences.

(1) Is $3/(n+5)$ increasing or decreasing?
Solution: For this case it's easiest to compare the n^{th} term and the $(n+1)^{\text{th}}$ term. To do this we simply plug in and we notice,

$$\frac{3}{n+5} > \frac{3}{(n+1)+5} = \frac{3}{n+6}.$$

Since this is true for all $n \geq 1$, the sequence is decreasing.

(2) Show $\left\{\frac{n}{n^2+1}\right\}_{n=1}^{\infty}$ is decreasing.
Solution: For this it's easier to take the derivative,

$$\left(\frac{n}{n^2+1}\right)' = \frac{1-n^2}{(n^2+1)^2} < 0; \text{ for } n > 1.$$

Therefore, the sequence is decreasing.

Definition 2. A sequence a_n is said to be bounded above if there is an M such that $a_n \leq M$ for all $n \geq 1$, and bounded below if there is an m such that $a_n \geq m$ for all $n \geq 1$.

Theorem 1. *Every bounded monotonic sequence is convergent.*

Difference Equations (aka Recurrence Relations, aka Recursive Formula).

These types of sequences come up often in various applications. The idea is that subsequent elements in the sequence will depend on previous elements in the sequence. We can think of the $(n+1)^{\text{th}}$ term as a function of a combination of other terms, i.e. $a_{n+1} = f(a_n, a_{n-1}, \ldots, a_1)$.

(1) Lets try to find the limit of the following difference equation: $a_{n+1} = (a_n+6)/2$. Notice that if the limit exists, $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = a_*$. Here, a_* is called the fixed point of the difference equation. Now, we can plug this in and find the value for it,

$$a_* = \frac{1}{2}(a_* + 6) \Rightarrow a_* = 6.$$

Examples From the Book.

46) Notice $\lim_{n\to\infty}\left|\frac{\sin^2 n}{2^n}\right| \leq \lim_{n\to\infty}\frac{1}{2^n} = 0 \Rightarrow \lim_{n\to\infty}\frac{\sin^2 n}{2^n} = 0$.

47) $\lim_{n\to\infty}\frac{n}{2^n} = \lim_{n\to\infty}\frac{n}{e^{n\ln 2}} = \lim_{n\to\infty}\frac{1}{\ln 2 e^{n\ln 2}} = 0$.

68) $\lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty}\frac{\ln(1+1/n)}{1/n} = \lim_{n\to\infty}\frac{-1/(n^2+n)}{-1/n^2} = \lim_{n\to\infty}\frac{n^2}{n^2+n} = \lim_{n\to\infty}\frac{1}{1+1/n} = 1$.

71) For this problem we must take $e^{\ln}$, then take the limit

$$\lim_{n\to\infty}\left(\frac{x^n}{2n+1}\right)^{1/n} = \lim_{n\to\infty} e^{\ln(x^n/(2n+1))^n} = e^{\lim_{n\to\infty}\ln(x^n/(2n+1))/n}.$$

Lets first compute the limit then plug it back in,

$$\lim_{n\to\infty}\frac{\ln(x^n/(2n+1))}{n} = \lim_{n\to\infty}\frac{\ln(x^n)-\ln(2n+1)}{n} = \lim_{n\to\infty}\frac{n\ln x-\ln(2n+1)}{n}$$
$$= \lim_{n\to\infty}\frac{\ln x - 2/(2n+1)}{1} = \lim_{n\to\infty}\ln x - \frac{2}{2n+1} = \ln x.$$

Plugging this back in gives, $e^{\ln x} = x$.

72) For this we must use our $e^{\ln}$ trick again,

$$\lim_{n\to\infty}\left(1-\frac{1}{n^2}\right)^n = \lim_{n\to\infty} e^{\ln(1-1/n^2)^n} = e^{\lim_{n\to\infty} n\ln(1-1/n^2)}.$$

So, lets look at the limit then plug it back in,

$$\lim_{n\to\infty}\frac{\ln(1-1/n^2)}{1/n} = \lim_{n\to\infty}\frac{-2/(n-n^3)}{-1/n^2} = \lim_{n\to\infty}\frac{2n^2}{n-n^3} = \lim_{n\to\infty}\frac{2}{1/n-n} = 0.$$

Then, plugging back in gives, $e^0 = 1$.

84) Once again,

$$\lim_{n\to\infty} e^{\ln(n^2+n)^{1/n}} = e^{\lim_{n\to\infty}\frac{1}{n}\ln(n^2+n)}.$$

Computing the limit gives,

$$\lim_{n\to\infty}\frac{\ln(n^2+n)}{n} = \lim_{n\to\infty}\frac{(2n+1)/(n^2+n)}{1} = \lim_{n\to\infty}\frac{2n+1}{n^2+n} = \lim_{n\to\infty}\frac{2+1/n}{n+1} = 0.$$

Plugging back in gives, $e^0 = 1$.

90) We have done a problem like this in the improper integrals section. This reiterates the intimate relationship between integrals and sequences. $\lim_{n\to\infty}\int_1^n \mathrm{d}x/x^p$, for $p > 1$ converges to $1/(p-1)$. We can see this by integrating it.

10.2 Series

A series is a sum of sequential terms. An infinite series can be represented as such: $\sum_{n=1}^{\infty} a_n$. We also think of series as a sequence of partial sums, where each partial sum is $s_N = \sum_{n=1}^{N} a_n$. We have to make sure we don't confuse these very different sequences. One is a sequence that is being summed, the other is a sequence of sums.

Definition 3. Given $\sum_{n=1}^{\infty} a_n$, let $s_n = \sum_{i=1}^{n} a_i$ bet the partial sums. If s_n converges and $\lim_{n\to\infty} s_n = s$ exists, then we say $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} a_n = s$. Otherwise, we say it diverges.

Ex: Consider the series $\sum_{n=1}^{\infty} ar^{n-1}$. This is a very important series called the geometric series. What does this converge to?

Notice if $r = 1$, $s_n = a + a + \cdots + a = na \to \pm\infty$, so it diverges. Now, if $r = -1$, the partial sum will jump between zero and one, so it also diverges.

If $|r| \neq 1$, $s_n = a + ar + ar^2 + \cdots + ar^{n-1}$, and $rs_n = ar + ar^2 + \cdots + ar^n$, then $s_n - rs_n = a - ar^n \Rightarrow s_n = \frac{a(1-r^n)}{1-r}$. Now, for $-1 < r < 1$, $r^n \to 0$ as $n \to \infty$, hence $\lim_{n\to\infty} s_n = a/(1-r)$. For $|r| > 1$, $r^n \to \infty$, so s_n clearly diverges

Theorem 2. *The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges, for $|r| < 1$ to*

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \tag{1}$$

and diverges otherwise.

(1) Find the sum of $S = 5 - 10/3 + 20/9 - 40/27 + \cdots$.
Solution: Notice that we can immediately factor out a 5, $S = 5[1 - 2/3 + 4/9 - 8/27 + \cdots]$. Now we notice that we have alternating sings, so we must have a $(-1)^{n-1}$ because the first term is positive (if the first term was negative it would be $(-1)^n$). Next, we notice that all the terms are powers of 2/3, via the geometric series theorem, our sum is

$$\sum_{n=1}^{\infty} 5\left(-\frac{2}{3}\right)^{n-1} = \frac{5}{1+2/3} = \frac{5}{5/3} = 3.$$

(2) Is $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ convergent or divergent?
Solution: This series isn't in the form of the geometric series, so we must convert it to that form,

$$\sum_{n=1}^{\infty} 2^{2n} 3^{1-n} = \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}.$$

This does not converge because $4/3 > 1$, so it violates the hypothesis of the geometric series theorem.

(3) Write $2.3\overline{17}$ as a geometric series.
Solution: We must think of this as a constant plus a fraction,

$$2.3\overline{17} = 2.3+\frac{17}{10^3}+\frac{17}{10^5}+\cdots = 2.3+17\left[\frac{1}{10^3}+\frac{1}{10^5}\cdots\right] = 2.3+17\sum_{n=1}^{\infty}\left(\frac{1}{10}\right)^{2n+1}.$$

(4) For what values of x does $\sum_{n=0}^{\infty} x^n$ (this is called a power series) converge?
Solution: This is exactly a geometric series if x were fixed. Now, we may not be able to see this right away, but if we play around with the index we see that

$$\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x};\ |x| < 1. \tag{2}$$

The next couple of examples are telescoping and harmonic series. These will illustrate some concepts that can easily be confused.

(1) Telescoping series: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.
Notice, this looks a lot like a partial fraction, so $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. So we get,

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\frac{1}{n}-\frac{1}{n+1} = \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right) = 1-\frac{1}{n+1} \to 1 \quad \text{as} \quad n\to\infty.$$

(2) Harmonic series: $\sum_{n=1}^{\infty} 1/n$.
Notice, that the sequence $1/n$ converges to 0 as $n \to \infty$, however we will show that the series diverges. In order to do this we calculate the partial sums and put estimates on them, $s_1 = 1$, $s_2 = 1 + 1/2$, $s_4 = 1 + 1/2 + (1/3 + 1/4) > 1 + 1/2 + (1/4 + 1/4) = 2$, $s_8 > 1 + 3/2$, $s_{16} > 1 + 4/2$, $s_{32} > 1 + 5/2$, $s_{64} > 1 + 6/2$. So, $s_{2^n} > 1 + n/2 \Rightarrow \infty$ as $n \to \infty$. So, by definition, the series diverges.

The following theorems give us a frame work to prove divergence but NOT convergence.

Theorem 3. *If the series* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

Proof. We can calculate the partial sums,

$$s_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$$
$$s_{n-1} = a_1 + a_2 + \cdots + a_{n-2} + a_{n-1}$$

Now, if we subtract the two, we get $s_n - s_{n-1} = a_n$, so we have a representation of a_n from the partial fractions. Now, since the series converges, the partial sums converge to exactly that sum, so $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} s_{n-1} = s$. Therefore, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - s_{n-1} = s - s = 0$. □

Corollary 1. *If* $\lim_{n\to\infty} a_n \neq 0$ *or doesn't exist, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Ex: Show $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

Solution: We can just show that the sequence a_n doesn't go to zero.

$$\lim_{n\to\infty} \frac{n^2}{5n^2+4} = \lim_{n\to\infty} \frac{1}{5+4/n^2} = \frac{1}{5}.$$

Here are some properties of sums that we should keep in mind,

Theorem 4. *If $\sum a_n$ and $\sum b_n$ converge, $\sum ca_n$ and $\sum a_n \pm b_n$ converge, and*

$$\text{(3)} \quad a) \sum ca_n = c\sum a_n \qquad \text{and} \qquad b) \sum(a_n \pm b_n) = \sum a_n \pm \sum b_n.$$

Ex: Does $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$ converge? If so, find the sum.

Solution: First we find the two sums individually,

$$\sum_{n=1}^{\infty} \frac{3}{n(n+1)} = 3\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = 3.$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} = \frac{1/2}{1-1/2} = 1.$$

So, the series converges to $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right) = 4.$