10.3 Integral test

I can't quite give the full motivation as I did in class, but basically if we look at an integral we can approximate it by a sum. Since integrals and sums are so intimately connected, we can make a conclusion about the sum from evaluating the integral.

Ex: Lets look at $\sum_{n=1}^{\infty} 1/n^2$. If we look at the partial sums we have $\lim_{n\to\infty} s_n < 1+\int_1^{\infty} \mathrm{d}x/x^2$ because the partial sums will just be right Riemann sums after the first one. So, if the integral converges the series will also converge. But we already know the integral converges since $p > 1$. So, the series too converges.

We have a similar result for series that diverge, but let's not go over that and get straight to the test. To see for yourself test it out with $\sum_{n=1}^{\infty} 1/n$.

Theorem 1. *Integral test: Suppose f is continuous, positive, and decreasing on $[1, \infty)$ and let $a_n = f(n)$, then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the integral $\int_1^{\infty} f(x)dx$ also converges, i.e.*

$$\int_1^{\infty} f(x)dx \text{ converges } \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges.} \tag{1}$$

$$\int_1^{\infty} f(x)dx \text{ diverges } \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges.} \tag{2}$$

(1) Test $\sum_{n=1}^{\infty} 1/(n^2+1)$
Solution: We integrate:

$$\int_1^{\infty} \frac{\mathrm{d}x}{x^2+1} = \lim_{t\to\infty} \int_1^{t} \frac{\mathrm{d}x}{x^2+1} = \lim_{t\to\infty} \tan^{-1} x|_1^t = \lim_{t\to\infty} (\tan^{-1} t - \pi/4) = \frac{\pi}{4}.$$

(2) For what values of p does $\sum_{n=1}^{\infty} 1/n^p$ converge?
Solution: We have to integrate $\int_1^{\infty} \mathrm{d}x/x^p$, but we already know this converges for $p > 1$, and by the integral test, the series too converges for $p > 1$ and diverges otherwise. This is called a p-series.

Theorem 2. *P-series: The series $\sum_{n=1}^{\infty} 1/n^p$ converges for $p > 1$, and diverges otherwise.*

(1) $\sum_{n=1}^{\infty} 1/n^3$ converges because $p = 3 > 1$.
(2) $\sum_{n=1}^{\infty} 1/n^{1/3}$ diverges because $p = 1/3 < 1$.
(3) Test $\sum_{n=1}^{\infty} (\ln n)/n$.
Solution: We have to integrate this,

$$\int_1^{\infty} \frac{\ln x}{x}\mathrm{d}x = \lim_{t\to\infty} \int_1^{t} \frac{\ln x}{x}\mathrm{d}x = \lim_{t\to\infty} \frac{1}{2}(\ln x)^2\Big|_1^t = \lim_{t\to\infty} \frac{1}{2}(\ln t)^2 = \infty$$

There will be times when we wont be able to find the sum of certain convergent series. In these cases it is beneficial to estimate the sum. Notice the bigger partial sum we take, the better the estimate, but how can we tell how good the estimate is? Since it converges, we can use two integrals to do this. Notice that $s \leq s_n + \int_n^\infty f(x)\mathrm{d}x$ and $s \geq s_n + \int_{n+1}^\infty f(x)\mathrm{d}x$ because these are like left and right hand Riemann estimates for integrals of monotonic functions.

Definition 1. Suppose $\sum_{n=1}^\infty a_n = s$, and s_n are it's partial sums. Then the <u>remainder</u> of the n^{th} partial sum is $R_n = s - s_n$.

Theorem 3. *Remainder: Consider* $\sum^\infty a_n = s$*. Suppose* $f(x) = a_k$*, where* f *is continuous, positive, and decreasing for* $x \geq n$*, then*

$$\int_{n+1}^\infty f(x)\,dx \leq R_n \leq \int_n^\infty f(x)\,dx. \tag{3}$$

Ex: Consider $\sum_{n=1}^\infty 1/n^3$.

(a) Find the maximum error for $n = 10$.
Solution: We just plug this into the formula,

$$R_{10} \leq \int_{10}^\infty \frac{\mathrm{d}x}{x^3} = \lim_{t\to\infty} \int_{10}^t \frac{\mathrm{d}x}{x^3} = \lim_{t\to\infty} \frac{-1}{2x^2}\bigg|_{10}^t = \lim_{t\to\infty} \frac{-1}{2t^2} - \frac{-1}{2*(10)^2} = \frac{1}{200} = .005.$$

(b) How many terms must we take for $R_n \leq .0005$?
Solution: Here we bound our formula and see what n has to be,

$$R_n \leq \int_n^\infty \frac{\mathrm{d}x}{x^3} = \frac{1}{2n^2} < .0005 \Rightarrow n^2 > \frac{1}{.001} = 1000 \Rightarrow n > \sqrt{1000} \approx 31.6.$$

So, we must take 32 terms.

(c) Now, notice if we add s_n to both sides of the inequality we get bounds on the exact solution, i.e. $s_10 \approx 1.1975$, so for $n = 10$.

$$\begin{aligned}
s_10 + \int_{11}^\infty f(x)\mathrm{d}x \leq R_10 + s_10 \leq s_10 + \int_{10}^\infty f(x)\mathrm{d}x \\
\Rightarrow 1.1975 + \frac{1}{242} \leq s \leq 1.1975 + \frac{1}{200} \\
\Rightarrow 1.2016 \leq \sum_{n=1}^\infty 1/n^3 \leq 1.2025.
\end{aligned}$$

10.4 Comparison tests

This is very similar to integral comparison tests.

Theorem 4. *Direct Comparison: Suppose $\sum a_n$ and $\sum b_n$ have positive terms, then*

(i) *If $\sum b_n$ converges and $a_n \leq b_n$ for all n, then $\sum a_n$ also converges.*
(ii) *If $\sum b_n$ diverges and $a_n \geq b_n$ for all n, then $\sum a_n$ also diverges.*

State whether the following converge or diverge, and state the reasoning.

(1) $\sum_{n=1}^{\infty} 1/(2^n + 1)$.
Solution: We know $\frac{1}{2^n+1} < \frac{1}{2^n}$. Therefore, since $\sum_{n=1}^{\infty} 1/2^n$ converges because of p-series, where $p > 1$, $\sum_{n=1}^{\infty} 1/(2^n + 1)$ also converges by the direct comparison test.

(2) $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$.
Solution: Here as usual we take the highest power of the top and bottom. This will give us $5/2n^2$. We know the sum of this converges, so for direct comparison we would attempt to show that this is greater than our original sequence. This is easy to show since all the terms in the denominator are additive, so $\frac{5}{2n^2+4n+3} \leq \frac{5}{2n^2}$. Since $\sum_{n=1}^{\infty} 1/n^2$ converges by p-series because $p > 1$, the original series also converges by the direct comparison test.

(3) $\sum_{n=1}^{\infty} (\ln n)/n$.
Solution: Since $\ln n > 1$ for $n \geq 3$, $\frac{\ln n}{n} \geq \frac{1}{n}$ for $n \geq 3$. Further, since $\sum_{n=1}^{\infty} 1/n$ diverges by p-series because $p = 1$, by the direct comparison test, the original series converges as well. Notice that we only care about the tail end.

Notice that we can't use this test on something like $\sum_{n=1}^{\infty} 1/(2^n - 1)$, so we need the limit comparison test,

Theorem 5. *Limit comparison: Suppose $\sum a_n$ and $\sum b_n$ have positive terms, and $\lim_{n\to\infty} a_n/b_n = c > 0$, where c is a finite number. Then, either both $\sum a_n$ and $\sum b_n$ converge or both diverge. Further, if $c = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges, and if $c = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.*

State whether the following converge or diverge, and state the reasoning.

(1) $\sum_{n=1}^{\infty} 1/(2^n - 1)$.
Solution: Again we take the highest power of both the top and the bottom, i.e. $1/2^n$. Taking the limit gives,

$$\lim_{n\to\infty} \frac{1/(2^n - 1)}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - 1/2^n} = 1 > 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges by geometric series because $|1/2| < 1$, by the limit comparison test $\sum_{n=1}^{\infty} 1/(2^n - 1)$ also converges.

(2) $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$.
Solution: As before we take the largest power of the numerator and largest part of the denominator, i.e. $2n^2/n^{5/2} = 2/\sqrt{n}$. Taking the limit gives,

$$\lim_{n\to\infty} \frac{(2n^2+3n)/\sqrt{5+n^5}}{2/\sqrt{n}} = \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{2\sqrt{5+n^5}} = \lim_{n\to\infty} \frac{2+3/n}{2\sqrt{5/n^5+1}} = 1.$$

Since $\sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges by p-series because $p < 1$, by the limit comparison test, $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ also diverges.

10.5 Ratio and Root Tests

Sometimes we need to bring out the big guns,

Theorem 6. *Ratio test: Consider $\sum a_n$, and suppose $\lim_{n\to\infty} |a_{n+1}/a_n| = L$, then*

a) *If $L < 1$, then $\sum a_n$ converges absolutely,*
b) *If $L > 1$, then $\sum a_n$ diverges,*
c) *and if $L = 1$, the test is inconclusive.*

State whether the following converge or diverge, and state the reasoning.

(1) $\sum_{n=1}^{\infty} n^3/3^n$.
Solution: We apply the ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}\right| = \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1+\frac{1}{n}\right)^3.$$

Taking the limit of this gives, $\lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^3 = \frac{1}{3} < 1$. Therefore, by the ratio test, $\sum_{n=1}^{\infty} n^3/3^n$ converges absolutely.

(2) $\sum_{n=1}^{\infty} n^n/n!$.
Solution: We apply the ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}\right| = \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n.$$

Taking the limit gives,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \exp\left[\lim_{n\to\infty} n\ln(1+\frac{1}{n})\right],$$

Now, we look at just the inside,

$$\lim_{n\to\infty} n\ln(1+\frac{1}{n}) = \lim_{n\to\infty} \frac{\ln(1+\frac{1}{n})}{1/n} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1.$$

then,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1.$$

Therefore, by the ratio test, $\sum_{n=1}^{\infty} n^n/n!$ diverges.

Theorem 7. *Root test: Consider* $\sum a_n$*, and suppose* $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L$*, then*

a) *If* $L < 1$*, then* $\sum a_n$ *converges absolutely,*
b) *If* $L > 1$*, then* $\sum a_n$ *diverges,*
c) *and if* $L = 1$*, the test is inconclusive.*

State whether the following converge or diverge, and state the reasoning.

(1) $\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n$.

Solution: We apply the root test,

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left|\left(\frac{2n+3}{3n+2}\right)^n\right|} = \frac{2n+3}{3n+2} = \frac{2+3/n}{3+2/n}.$$

Taking the limit gives, $\lim_{n\to\infty}\frac{2+3/n}{3+2/n} = \frac{2}{3} < 1$. Therefore, by the root test, $\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n$ converges absolutely.