Exam III Fall 2016:

(1) Converges by GST since $|r| = 1/3 < 1$ and

$$\sum_{n=2}^{\infty} 5\left(\frac{-1}{3}\right)^n = \sum_{n=0}^{\infty} 5\left(\frac{-1}{3}\right)^{n+2} = \sum_{n=0}^{\infty} \frac{5}{9}\left(\frac{-1}{3}\right)^n = \frac{5/9}{1+1/3} = \frac{3}{4}\cdot\frac{5}{9} = \frac{5}{12}$$

(2) (a) Converges by GST since $|r| = 3/4 < 1$ because

$$\sum_{n=0}^{\infty} \frac{3^{n+1}}{2^{2n}} = \sum_{n=0}^{\infty} 3\frac{3^n}{4^n} = 3\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n = \frac{3}{1-3/4} = 12.$$

(b) This diverges by the nth term test since

$$\lim_{n\to\infty} \frac{n^2}{(n+2)(n-2)} = \lim_{n\to\infty} \frac{1}{(1+2/n)(1-2/n)} = 1 \neq 0$$

(3) (a) We compare to $1/n^{1/3}$,

$$\lim_{n\to\infty} \frac{2n^2/\sqrt[3]{n^7+n}}{1/n^{1/3}} = \lim_{n\to\infty} \frac{2n^{7/3}}{\sqrt[3]{n^7+n}} = \lim_{n\to\infty} \frac{2}{\sqrt[3]{1+1/n^6}} = 2\checkmark$$

$\sum_{n=1}^{\infty} 1/n^{1/3}$ diverges by p-test since $p < 1$, so the original series also diverges by LCT.

(b) Notice $n/(e^{-n}+n^3) \leq 1/n^2$ and $\sum_{n=1}^{\infty} 1/n^2$ converges by p-test since $p > 1$, so the original series converges by DCT.

(4) This is one of the few problems where we have to use integral test,

$$\lim_{t\to\infty} \int_2^t \frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty} \int_2^t \frac{du}{\sqrt{u}} = \lim_{t\to\infty} 2u^{1/2}\Big|_2^t = \lim_{t\to\infty} 2\sqrt{\ln x}\Big|_2^t = \lim_{t\to\infty} 2\sqrt{\ln t} - 2\sqrt{\ln 2} = \infty$$

So the absolute value diverges by integral test. But now we have to test for conditional convergence of the original series by AST. Notice $\lim_{n\to\infty} 1/n\sqrt{\ln n} = 0\checkmark$ and $1/((n+1)\sqrt{\ln(n+1)}) \leq 1/n\sqrt{\ln n}\checkmark$, therefore the original series converges conditionally by AST.

(5) This converges absolutely since $|f(n)| = f(n) > 0$.

(6) Here we use ratio test,

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \left|\frac{(x-1)^{n+1}}{(n+1)3^{n+1}}\cdot\frac{n3^n}{(x-1)^n}\right| = \lim_{n\to\infty} \frac{1}{3}\left(\frac{n}{n+1}\right)|x-1| = \lim_{n\to\infty} \frac{1}{3}\left(\frac{1}{1+1/n}\right)|x-1| = \frac{1}{3}|x-1| < 1$$

Hence $|x-1| < 3 = R$ and the interval of absolute convergence is $-2 < x < 4$. If $x = 4$, $\sum_{n=1}^{\infty}(x-1)^n/(n3^n) = \sum_{n=1}^{\infty} 1/n$ diverges by p-test since $p = 1$. If $x = -2$, $\sum_{n=1}^{\infty}(x-1)^n/(n3^n) = \sum_{n=1}^{\infty}(-1)^n/n$ converges conditionally by AST since $\lim_{n\to\infty} 1/n = 0\checkmark$ and $1/(n+1) \leq 1/n$ and $\sum_{n=1}^{\infty} |(-1)^n/n|$ diverges as shown above. Therefore the radius of convergence is $R = 3$ and the interval of convergence is $x \in [-2, 4)$.

(7) (a) We first find the polynomial of order 3. $f(3) = 0$, $f'(3) = (x-2)^{-1}\big|_{x=3} = 1$, $f''(3) = -(x-2)^{-2}\big|_{x=3} = -1$, and $f'''(3) = 2(x-2)^{-3}\big|_{x=3} = 2$, so

$$P_3(x) = (x-3) - \frac{1}{2}(x-3)^2 + \frac{1}{3}(x-3)^3.$$

(b) We look for the pattern in the nth derivative, $f^{(n)}(3) = (-1)^{n+1}(n-1)!$, so we get

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n-1}(x-3)^{n-1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}(x-3)^n$$

(8) (a) Here we can see that we get a factor of $-1/2$ at each derivative so, $f^{(n)}(1/2) = (-1/2)^n e^{-1/2}$ then our Taylor series is

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2}\right)^n \frac{e^{-1/2}}{n!}(x-1)^n$$

(b) For the radius of convergence we use ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1/2)^{n+1}e^{-1/2}(x-1)^{n+1}/(n+1)!}{(-1/2)^n e^{-1/2}(x-1)^n/n!}\right| = \lim_{n\to\infty} \frac{|x-1|}{2(n+1)} = 0$$

Therefore the radius of convergence is $R = \infty$ and the interval of convergence is $x \in (-\infty, \infty)$

(9) We use the common Taylor series,

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}$$

(10) Now we integrate the above,

$$\int \cos(x^2)dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(2n)!(4n+1)} + C$$

(11) Now we need to find the remainder in order to get the proper approximation error,

$$|R_n(x)| \leq \frac{x^{4n+5}}{(2n+2)!(4n+5)} \Rightarrow \max(|R_n(x)|) \leq \frac{1}{(2n+2)!(4n+5)} \leq \frac{1}{100}$$

Since this cannot be solved easily, we just plug in values of n until we get the sufficient error. $n = 0 \Rightarrow |\text{Error}| \leq 1/10$ and $n = 1 \Rightarrow |\text{Error}| \leq 1/(24 \cdot 9) < 1/100$. So, $n = 1$ works, i.e.

$$\int \cos(x^2)dx \approx x - \frac{x^5}{10}$$