(1) Here we employ ratio test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{3^{n+1}\sqrt{(n+1)^2+100}}\cdot\frac{3^n\sqrt{n^2+100}}{x^n}\right| = \sqrt{\frac{n^2+100}{(n+1)^2+100}}\frac{1}{3}|x| = \lim_{n\to\infty}\frac{1}{3}\sqrt{\frac{1+100/n^2}{(1+1/n)^2+100/n^2}}|x| = \frac{1}{3}|x| < 1$$

Then the radius of convergence is $\boxed{R=3}$.

If $x=3$, $\sum_{n=0}^{\infty}\frac{x^n}{3^n\sqrt{n^2+100}} = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n^2+100}}$ and

$$\lim_{n\to\infty}\frac{1/\sqrt{n^2+100}}{1/n} = \lim_{n\to\infty}\frac{n}{\sqrt{n^2+100}}\lim_{n\to\infty}\frac{1}{\sqrt{1+100/n^2}} = 1\checkmark$$

$\sum_{n=1}^{\infty} 1/n$ diverges by p-test because $p=1$, so the power series <u>diverges</u> at $x=3$ by LCT.

If $x=-3$, $\sum_{n=0}^{\infty}\frac{x^n}{3^n\sqrt{n^2+100}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n^2+100}}$, $\lim_{n\to\infty} 1/\sqrt{n^2+100} = 0\checkmark$ and $1/\sqrt{(n+1)^2+100} \le 1/\sqrt{n^2+100}\checkmark$, so the power series <u>converges conditionally</u> at $x=-3$ by AST.

Therefore the interval of convergence is $\boxed{-3 < x \le 3}$.

(2) (a) The sum <u>converges by GST</u> since $|r| = 1/4 < 1$, and the sum is $\frac{1/2}{1+1/4} = 2/5$.
(b) The sum <u>diverges by the nth term test</u> since $\lim_{n\to\infty} 1/(10+e^{2/n}) = 1/11 \ne 0$.

(3) We use LCT to test absolute convergence

$$\lim_{n\to\infty}\frac{1/(2n+3)}{1/n} = \lim_{n\to\infty}\frac{n}{2n+3} = \lim_{n\to\infty}\frac{1}{2+3/n} = 1/2\checkmark$$

$\sum_{n=1}^{\infty}$ <u>diverges</u> by p-test since $p=1$, so the original series cannot be absolutely convergence.

Now lets test conditional convergence, $\lim_{n\to\infty} 1/(2n+3) = 0\checkmark$ and $1/(2(n+1)+3) \le 1/(2n+3)\checkmark$. So the series is <u>conditionally convergent by AST</u>.

(4) The series is convergence since ratio test implies absolute convergence.

(5) (a) Taking the first three derivatives gives us $f(\pi/4) = \sqrt{2}/2$, $f'(\pi/4) = \sqrt{2}/2$, $f''(\pi/4) = -\sqrt{2}/2$, and $f'''(\pi/4) = -\sqrt{2}/2$, then

$$P_3(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x-\frac{\pi}{4}\right)^3$$

(b) Notice that this is not an alternating series and that we are given a point instead of an interval so in order to take the remainder we need to bound the fourth derivative,

$$f^{(4)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \Rightarrow |R_3(x)| \le \frac{\sqrt{2}}{2\cdot 4!}|x-\frac{\pi}{4}|^4 \Rightarrow \left|R_3\left(\frac{\pi}{12}\right)\right| \le \frac{\sqrt{2}}{2\cdot 4!}\left(\frac{\pi}{6}\right)^4 = \frac{\sqrt{2}\pi^4}{48\cdot 6^4} \Rightarrow \text{(b)}$$

(6) Lets try to get what the nth derivative is, $f(3) = 1/9$, $f'(x) = -2x^{-3}$, $\ldots$, $f^{(n)}(x) = (-1)^n(n+1)!x^{-(n+2)}$, so $f^{(n)}(3) = (-1)^n(n+1)!3^{-(n+2)}$ so the Taylor series is

$$\sum_{n=0}^{\infty}(-1)\frac{(n+1)!3^{-(n+2)}}{n!}(x-3)^n = \sum_{n=0}^{\infty}(-1)^n(n+1)3^{-(n+2)}(x-3)^n$$

(7) (a) We know what sine is, so

$$f(x) = x\sum_{n=0}^{\infty}(-1)^n\frac{(3x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{3^{2n+1}x^{2n+2}}{(2n+1)!}$$

(b) Now we use ratio test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{2n+3}x^{2n+4}}{(2n+3)!}\cdot\frac{(2n+1)!}{3^{2n+1}x^{2n+2}}\right| = \lim_{n\to\infty}\frac{9}{(2n+3)(2n+4)}|x|^2 = 0$$

So the radius of convergence is $\boxed{R=\infty}$ and the interval of convergence is $\boxed{-\infty < x < \infty}$.

(8) This is an example where LCT doesn't work. We use DCT,

$$\frac{2}{n^{3/2}} \le \frac{|\cos^2 n + 2|}{n^{3/2}} \le \frac{3}{n^{3/2}}$$

Since $\sum_{n=1}^{\infty} 1/n^{3/2}$ <u>converges</u> by p-test since $p > 1$, we use the RHS, and this to converges by p-test. So, by DCT the original series also <u>converges</u>.

(9) It's a polynomial, so $f(x) = x - x^k/k!$.