Basic Integration Review

Here we will review integration techniques from section 5.4 to 5.6.
Sec. 5.4:

23) We simplify the expression and integrate it straight,

$$\int_1^{\sqrt{2}} \frac{s^2+\sqrt{s}}{s^2} ds = \int_1^{\sqrt{2}} \left(1+s^{-3/2}\right) ds = s - 2s^{-1/2}\Big|_1^{\sqrt{2}} = \sqrt{2} - \frac{2}{2^{1/4}} + 1.$$

Sec. 5.5:

(1) Here we make a u-sub, $u = 7-3y^2 \Rightarrow du = -6ydy$,

$$\int 3y\sqrt{7-3y^2}dy = \int -2u^{1/2}du = -\frac{4}{3}u^{3/2} + C = -\frac{4}{3}(7-3y^2)^{3/2} + C.$$

(2) Again, we make a u-sub, $u = 1/t - 1 \Rightarrow du = -dt/t^2$,

$$\int \frac{1}{t^2}\cos\left(\frac{1}{t}-1\right)dt = -\int \cos u du = -\sin u + C = -\sin\left(\frac{1}{t}-1\right) + C.$$

(3) Here we have to recognize a familiar derivative,

$$\int \frac{5}{9+4r^2}dr = \frac{5}{9}\int \frac{dr}{1+\frac{4}{9}r^2} = \frac{5}{9}\tan^{-1}\left(\frac{2}{3}r\right) + C.$$

Sec. 5.6:

27) The u-sub is $u = 2-\cos t \Rightarrow du = \sin t dt$,

$$\int_0^{\pi} \frac{\sin t}{2-\cos t}dt = \int_1^3 \frac{du}{u} = \ln u\Big|_1^3 = \ln 3.$$

6.1 Volumes using Cross-sections

The volume of any solid with cross-sectional area $A(x)$ from $x = a$ to $x = b$ is

$$V = \int_a^b A(x)dx \tag{1}$$

Disk Method

Specifically, the volume for rotations using disks (i.e. regions without gaps) about the x and y axes are respectively,

$$V = \int_a^b \pi R(x)^2 dx \tag{2}$$

$$V = \int_\alpha^\beta \pi R(y)^2 dy \tag{3}$$

Lets remind ourselves of some problems we did in class,

Ex: Find the volume of a sphere centered at the origin with radius r.

Solution: Consider circular cross-sections, i.e. circles cutting the sphere perpendicular to the x-axis of radius y. Then, by the distance formula (Pythagorean Theorem), $R(x)^2 = y^2 = r^2 - x^2$. So,

$$V = \int_a^b \pi R(x)^2 dx = \int_{-r}^{r} \pi(r^2-x^2)dx = \frac{4}{3}\pi r^3.$$

9) For this problem the radius is $R(y) = \sqrt{5}y^2/2$, then the cross-sectional area is $A = 5\pi y^4/4$, and the volume is

$$V = \pi \int_0^2 \frac{5}{4} y^4 dy = \frac{\pi}{4} y^5 \Big|_0^2 = 8\pi$$

Ex: Find the volume of a square pyramid such that the sides of the square are of length L and height h.

Solution: There are many ways to do this, but perhaps the easiest is to consider a triangle with the head at the origin and the base at $x = L$ with square cross-sections perpendicular to the x-axis. So, at an arbitrary x, the cross-section is a square of length, say l. Notice, the entire triangle and a triangle at any arbitrary x will be similar triangles, and hence have the same ratios. Therefore, $x/h = l/L$, i.e. the ratio of the heights equal the ratio of the lengths. This gives $l = Lx/h$. Then $A(x) = (Lx/h)^2$, and

$$V = \int_0^h \frac{L^2}{h^2} x^2 dx = \frac{1}{3} L^2 h$$

17) For this problem the radius is $R(y) = \tan(\pi y/4)$, so the area is $A = \pi \tan^2(\pi y/4)$, then $V = \pi \int_0^1 \tan^2\left(\frac{\pi}{4} y\right) dy$. Here we use the u-sub, $u = \pi y/4 \Rightarrow du = (\pi/4)dy$, then

$$V = 4 \int_0^{\pi/4} \tan^2 u du = 4 \int_0^{\pi/4} (\sec^2 u - 1) du = 4[\tan u - u]_0^{\pi/4} = 4 - \pi$$

31) The radius is $R(y) = \sqrt{5}y^2$, then the area is $A = \pi \cdot 5y^4$, then

$$V = 5\pi \int_{-1}^1 y^4 dy = 10\pi \int_0^1 y^4 dy = 2\pi y^5 \Big|_0^1 = 2\pi$$

Ex: Find the volume of a region bounded by $y = \sqrt{x}$, $x = 0$, and $x = 1$ about the x-axis.

Solution: The radius is $R(x) = \sqrt{x}$, so the area is $A = \pi x$, then

$$V = \int_0^1 \pi x dx = \frac{\pi}{2}.$$

Ex: Find the volume of a region bounded by $y = x^3$, $x = 0$, and $y = 8$ about the y-axis.

Solution: Since we are revolving around the y-axis we need to solve for x as a function of y, i.e. $x = y^{1/3}$. The radius will be $R(y) = y^{1/3}$, then the area is $A = \pi y^{2/3}$. This gives us

$$V = \int_0^8 \pi y^{2/3} dy = \frac{96}{5}\pi$$

Washer Method

The volume for rotations using washers (i.e. regions with gaps) about the x and y axes are respectively,

$$V = \int_a^b \pi \left[R(x)^2 - r(x)^2\right] dx \tag{4}$$

$$V = \int_\alpha^\beta \pi \left[R(y)^2 - r(y)^2\right] dy \tag{5}$$

where R is the radius of the larger region, and r is the radius of the smaller region (i.e. the gap).

Here are some problems we did in class,

41) Since this problem is more involved, in order to sketch the region we need to find where the two curves intersect. This is done by equating them,

$$x^2 + 1 = x + 3 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0 \Rightarrow x = 2, -1$$

So the points of intersection are $(-1, 2)$ and $(2, 5)$. From the sketch we see that $y = x + 3$ is furthest from the line of rotation (x-axis), so the big radius is $R(x) = x + 3$, and the little radius is $r(x) = x^2 + 1$, then our area becomes

$$A = \pi R(x)^2 - \pi r(x)^2 = \pi\left[(x+3)^2 - (x^2+1)^2\right] = \pi\left[-x^4 - x^2 + 6x + 8\right].$$

Then the volume is

$$V = \pi \int_{-1}^{2} \left(-x^4 - x^2 + 6x + 8\right) dx = \pi\left[-\frac{1}{5}x^5 - \frac{1}{3}x^3 + 3x^2 + 8x\right]_{-1}^{2} = \frac{117}{5}\pi$$

47) Here the outside radius is $R(y) = 2$ and the inside radius is $r(y) = \sqrt{y}$ giving an area of $A = \pi R(y)^2 - \pi r(y)^2 = \pi[4 - y]$. Then our volume is

$$V = \pi \int_0^4 (4 - y)dy = \pi\left[4y - \frac{1}{2}y^2\right]_0^4 = 8\pi.$$

49) The big radius is $R(y) = 2$ and the little radius is $r(y) = 1 + \sqrt{y}$. Then our area is $A = \pi R(y)^2 - \pi r(y)^2 = \pi[3 - 2\sqrt{y} - y]$. Then the volume is

$$V = \pi \int_0^1 \left(3 - 2y^{1/2} - y\right) dy = \pi\left[3y - \frac{4}{3}y^{3/2} - \frac{1}{2}y^2\right]_0^1 = \frac{7}{6}\pi$$

56) This problem has two parts. First we find the volume, then we find the rate of change of the water level.

(a) The radius is $R(y) = \sqrt{2y}$, which gives us $A = 2\pi y$, then the volume is

$$V = 2\pi \int_0^5 y dy = \pi y^2\Big|_0^5 = 25\pi.$$

(b) Now, for the related rates we need a general formula for the volume, which we actually found in the previous part: $V(y) = \pi y^2$, then we try to find dy/dt which gives us the rate of change of the water level,

$$\frac{dV}{dt} = 2\pi y \frac{dy}{dt} \Rightarrow \frac{dy}{dt} = \frac{dV/dt}{2\pi y} \Rightarrow \frac{dy}{dt}\Big|_{y=4} = \frac{3}{8\pi}.$$

Additional Examples (not in the book):

(1) Find the volume of the region between $y = x$ and $y = x^2$ revolved about the x-axis.

Solution: The points of intersection are $(0, 0)$ and $(1, 1)$. The larger radius is $R(x) = x$ and the smaller radius is $r(x) = x^2$. So the area is $A = \pi[x^2 - x^4]$, so the volume is

$$V = \int_0^1 \pi[x^2 - x^4]dx = \frac{2\pi}{15}.$$

(2) What if we revolve this around $y = 2$?

Solution: Since we are revolving about a line above our region of interest, our radii become $R(x) = 2 - x^2$ and $r(x) = 2 - x$. Then our area is $A = \pi\left[(2-x^2)^2 - (2-x)^2\right]$, so

$$V = \int_0^1 \pi\left[(2-x^2)^2 - (2-x)^2\right] dx = \frac{8\pi}{15}.$$

(3) Now, lets revolve the same region around $x = -1$.

Solution: First we must solve for x as a function of y. We get $x = y$ and $x = \sqrt{y}$, respectively. Since we are revolving about a line to the left of our region, the radii are $R(y) = \sqrt{y} - (-1)$ and $r(y) = y - (-1)$. So, the area is $A = \pi\left[(1+\sqrt{y})^2 - (1+y)^2\right]$ Then,

$$V = \int_0^1 \pi\left[(1+\sqrt{y})^2 - (1+y)^2\right] dy = \frac{\pi}{2}.$$

Tougher Examples (not in the book):

(1) Find the volume of a region such that the base is a circle centered at the origin of radius $r = 1$ and cross-sections perpendicular to the x-axis that are equilateral triangles.

Solution: Notice the base of the triangle will be $2y$, and since they are equilateral triangles we can split the triangle in half to get a "30, 60, 90" triangle with base y, so the height of the triangles will be $\sqrt{3}y$. This gives us a cross-sectional area of $A(y) = \sqrt{3}y^2$, but we have cross-sections perpendicular to the x-axis, so we need $A(x)$. Notice a circle is given by the equation $y^2 + x^2 = r^2$, but $r = 1$, so $y^2 = 1 - x^2$. Hence, $A(x) = \sqrt{3}(1 - x^2)$. Then

$$V = \int_{-1}^1 \sqrt{3}(1 - x^2)dx = \frac{4\sqrt{3}}{3}.$$

(2) Find the volume of a region such that the base is a semicircle centered at the origin with radius $r = 4$ and cross-sections perpendicular to the x-axis that are "30, 60, 90" triangles, where $30°$ is with respect to the x-axis.

Solution: Notice the base at any arbitrary x will be of length y. The height of the triangle will be $y/\sqrt{3}$, then $A(y) = y^2/2\sqrt{3}$. Since $r = 4$, $y^2 = 16 - x^2$. Then, $A(x) = (16 - x^2)/2\sqrt{3}$. So,

$$V = \int_{-4}^4 \frac{1}{2\sqrt{3}}(16 - x^2)dx = \frac{128}{3\sqrt{3}}.$$