

10.8 - 10.10 TAYLOR SERIES

Suppose the function f has the following power series:

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \dots = \sum_{n=0}^{\infty} c_n(x - a)^n. \tag{1}$$

Can we figure out what the coefficients are? Yes, yes we can. Notice that $f(a) = c_0$, so that gives us the first coefficient. For the second one lets differentiate to get $f'(x) = c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + \dots$. Now, if we plug in a we get $f'(a) = c_1$. How about the third? Well, $f''(x) = 2c_2 + 6c_3(x - a) + \dots$, so $f''(a) = 2c_2$. Can we figure out what c_n should be? Well we see that if we keep taking derivatives and evaluating them at the center, we get $f^{(n)}(x) = n!c_n + \dots$, so $c_n = f^{(n)}(x)/n!$. We have just derived a general formula for finding the coefficients of our series.

Theorem 1. Suppose $f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n$ converges for $|x - a| < R$. Then, $c_n = \frac{f^{(n)}(a)}{n!}$ and $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n$.

Definition 1. The series representation

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \dots \tag{2}$$

is called a Taylor series of f at $x = a$. If $a = 0$ we simply call this the Taylor series of f at $x = 0$ or the McLaurin series of f - both are used interchangeably.

Ex: Find the Taylor series of $f(x) = e^x$ and its radius of convergence.

Solution: This is easy because we can find the n^{th} derivative of e^x straightaway, i.e. $f^{(n)}(x) = e^x$, hence $f^{(n)}(0) = 1$. So $e^x = \sum_{n=0}^{\infty} x^n/n!$. Now, this is still a power series so like any other power series we can find the radius of convergence by using either root or ratio test. Lets apply ratio test,

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \frac{|x|}{n+1}.$$

Taking the limit gives us $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = 0$, so $R = \infty$. Therefore, the Taylor series converges everywhere and it is an exact representation of e^x .

Definition 2. Let $\xi \in (a, b)$, and f have k derivatives on (a, b) , then for any $n < k$ positive,

$$P_n(x) = f(\xi) + f'(\xi)(x - \xi) + \frac{f''(\xi)}{2}(x - \xi)^2 + \dots + \frac{f^{(n)}(\xi)}{n!}(x - \xi)^n$$

is the n^{th} order Taylor polynomial of f at $x = \xi$.

Notice that the Taylor polynomial is just the truncated Taylor series.

Lets do some problems that we did in class

10.8.7) We first evaluate the derivatives up to order $n = 3$: $f(\pi/4) = 1/\sqrt{2}$, $f'(\pi/4) = 1/\sqrt{2}$, $f''(\pi/4) = -1/\sqrt{2}$, and $f'''(\pi/4) = -1/\sqrt{2}$. Then the Taylor polynomial is

$$f(x) \approx P_3(x) = \frac{1}{\sqrt{2}} \left[1 + \left(x - \frac{\pi}{4}\right) - \frac{1}{2} \left(x - \frac{\pi}{4}\right)^2 - \frac{1}{6} \left(x - \frac{\pi}{4}\right)^3 \right].$$

10.8.13) We just convert this into a power series,

$$f(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n.$$

10.8.11) Here we just use the common Taylor series,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}.$$

10.8.19) For this problem we converted $\cosh x = (e^x + e^{-x})/2$, then we can use the Taylor series for e^x that we derived in the first example,

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2} \left[\sum_{n=0}^{\infty} \frac{x^n}{n!} + (-1)^n \frac{x^n}{n!} \right] = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

10.8.29) Here we have a case where the Taylor series is about a point other than zero.

$$f^{(n)}(2) = e^x \Big|_{x=2} = e^2 \Rightarrow e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!} (x-2)^n.$$

Ex: The derivatives of sine at $x = 0$ are as follows,

$$f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -1$$

If we continue this we see the pattern gives us

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \quad (3)$$

Ex: Similarly cosine has the following derivatives at $x = 0$,

$$f(0) = 1, f'(0) = 0, f''(0) = -1, f'''(0) = 0$$

If we continue this we see the pattern gives us

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \quad (4)$$

10.8.17) For this we just plug in $-x$ into the cosine equation above

$$7 \cos(-x) = 7 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

10.8.33) We want up to order 2 so lets find the respective polynomials up to order 2 and then just subtract them,

$$f(x) = \cos x - \frac{2}{1-x} = \left(1 - \frac{x^2}{2} + \dots \right) - (2 + 2x - 2x^2 + \dots) = -1 - 2x - \frac{5x^2}{2} + \dots$$

10.8.35) If we differentiate we notice

$$\frac{d}{dx} \ln(1+x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n; |x| < 1$$

To get back to $\ln(1+x)$ all we do is integrate the RHS,

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}; |x| < 1.$$

This means

$$(\sin x)(\ln(1+x)) = \left(x - \frac{x^3}{3!} + \dots \right) \left(x - \frac{x^2}{2} + \frac{x^3}{3!} + \dots \right) = \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^4}{6} + \dots \right) \approx x^2 - \frac{x^3}{2} + \frac{x^4}{6}; |x| < 1.$$

Since Taylor series are just power series, we will have the usual convergence results and remainders.

Theorem 2 (Taylor). *If f has derivatives up to order n at a neighborhood of $x = a$, then*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x) \quad (5)$$

where

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1}$$

for some ξ between x and a .

The fact that we have a remainder is cool and all, but we really need a way to bound it.

Theorem 3 (Remainder). *If $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$, then*

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1} \quad (6)$$

Lets do one remainder example before going back to it at the end,

Ex: (a) Find the series representation of $\int e^{-x^2} dx$.

Solution: We know what the Taylor series of e^x is about $x = 0$, so

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

We can integrate this term-by-term,

$$\int e^{-x^2} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(2n+1)}$$

(b) Find the value of n that approximates $\int_0^1 e^{-x^2}$ to 0.001.

Solution: Here we use the alternating series remainder,

$$|R_n(x)| \leq \frac{|x|^{2n+3}}{(n+1)!(2n+3)} \Rightarrow \max(|R_n(x)|) \leq \frac{1}{(n+1)!(2n+3)} < \frac{1}{1000}$$

and we'll see that $n = 4$ does the trick,

$$\int_0^1 e^{-x^2} dx \approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \approx 0.7475$$

Notice that there are some Taylor series that are extremely common and useful,

Common Taylor Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots \quad (7)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots \quad (8)$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \quad (9)$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \dots \quad (10)$$

Now lets do a bunch of assorted problems that we did in class,

10.9.11) If we have a single term multiplying a series, we can simply multiply through,

$$xe^x = x \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}.$$

Ex: Similarly,

$$x \cos x = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!}$$

10.9.10) Here we use the geometric series technique to derive a power series,

$$\frac{1}{2-x} = \frac{1/2}{1-x/2} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$$

10.9.29) Note that this technique tends to get convoluted, so if you don't feel comfortable using it then you should just take the derivatives and derive the Taylor series as you usually would. But lets try getting the first few terms of a Taylor series via multiplication

$$e^x \sin x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right) \left(x - \frac{x^3}{6} + \dots\right) = x + x^2 + \frac{1}{3}x^3 + \dots$$

Ex: This next example will show how to use Taylor series to simplify limits with indeterminate form,

$$\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \rightarrow 0} \frac{(1 + x + x^2/2 + x^3/6 + \dots) - 1 - x}{x^2} = \lim_{x \rightarrow 0} \frac{1}{2} + \frac{1}{6}x + \dots = \frac{1}{2}.$$

Ex: We could also do long division to find the first few terms of series, but that's not recommended at all,

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - x^3/6 + x^5/5! + \dots}{1 - x^2/2 + x^4/4! + \dots} = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$$

Ex: Find the first two terms of $f(x) = x \sin^2 x$.

Solution: Lets take derivatives and plug in the center, $f(0) = 0$,

$$f'(0) = \sin^2 x + 2x \sin x \cos x \Big|_{x=0} = 0,$$

$$f''(0) = 4 \sin x \cos x + 2x \cos^2 x - 2x \sin^2 x \Big|_{x=0} = 0,$$

$$f'''(0) = 6 \cos^2 x - 6 \sin^2 x - 8x \cos x \sin x \Big|_{x=0} = 6,$$

$$f^{(4)} = -32 \cos x \sin x + 8x \sin^2 x - 8x \cos^2 x \Big|_{x=0} = 0,$$

$$f^{(5)} = 40 \sin^2 x - 40 \cos^2 x + 32x \sin x \cos x \Big|_{x=0} = -40$$

Then we plug into the formula for Taylor series to get

$$x \sin^2 x \approx \frac{6}{3!}x^3 - \frac{40}{5!}x^5 = x^3 - \frac{1}{3}x^5$$

10.10.27) (a) This is a bit tricky for an exam, but there are a lot of pieces that will be useful for exam type problems. The original problem is difficult, but we notice that if we take two derivatives we get a geometric series like sum. Then we can work backwards to get our series.

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \Rightarrow \tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \Rightarrow \int_0^x \tan^{-1} x dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)(2n+2)}.$$

Now we look at the remainder and bound it,

$$|R_n(x)| \leq \frac{|x|^{2n+4}}{(2n+3)(2n+4)} \Rightarrow \max(|R_n(x)|) \leq \frac{1}{(2n+3)(2n+4)} < \frac{1}{1000}$$

We did this on the calculator and got $n = 15$ that satisfies the above condition.

(b) For this the maximum changes since the domain changes.

$$|R_n(x)| \leq \frac{(0.5)^{2n+4}}{(2n+3)(2n+4)} < \frac{1}{1000} \Rightarrow n = 2.$$

10.10.47) Here we find the sum of the series using geometric series,

$$x^3 + x^4 + x^5 + x^6 + \dots = \sum_{n=0}^{\infty} x^{n+3} = x^3 \sum_{n=0}^{\infty} x^n = \frac{x^3}{1-x}.$$

10.10.49) As we did above,

$$x^3 - x^5 + x^7 - x^9 + x^{11} + \dots = \sum_{n=0}^{\infty} (-1)^n x^{2n+3} = x^3 \sum_{n=0}^{\infty} (-x^2)^n = \frac{x^3}{1+x^2}.$$

10.9.35) We first write down the Taylor series of sine and then use the remainder formula to find the maximum error for our point.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \Rightarrow |R_n(x)| \leq \frac{|x|^5}{5!} \Rightarrow |R_1(0.1)| \leq \frac{(0.1)^5}{5!}$$

10.10.41) Here instead of a point we have a domain to find our maximum error in,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow |R_2(x)| \leq \frac{f^{(3)}(x)|x|^3}{3!} = \frac{e^x|x|^3}{6} \Rightarrow |R_2(x)| \leq \frac{e^{0.1}(0.1)^3}{6}$$

on the domain $|x| < 0.1$