6.5 WORK

Work is the sum of forces exerted over a certain distance. It's induced by an action. For a constant force, it will be $W = Fd$, where W is the work, F is the force or equivalently the weight, and d is the distance traversed. As we saw in class the problems get quite difficult, but the concept is fairly simple: force times distance. We can basically get two types of problems: 1) The force is found as a function (such as springs) or 2) We calculate an infinitesimal amount of work and integrate (such as ropes and tanks).

We first did some simple examples

Ex: How much work does it take to lift a 1.2 Kg book 0.7m.
Solution: Assuming $g = 10m/s^2$, the force is $F = 12$N, then the work is $W = Fd = (12)(0.7) =$ 8.4J.

Ex: How much work does it take to lift a 6lb weight 6ft?
Solution: Here we are already given the weight, which is equivalent to the force it exerts, so $W = Fd = 6 \cdot 6 = 36$ft-lb.

Now, what if the force changes as a function of distance? Then we need to add up all parts of the force, so we get the equation $W = \int_a^b F(x)dx$. For springs we recall Hooke's law states that a mass on a spring feels a force proportional to the length that the mass is stretched from the spring's natural position (i.e. $F = kx$), where x is the distance from the natural position and k is the spring constant.

4) Since we know $F = kx$, for a particular force at a particular length we have $k = F/x = (90N)/(1m) = 90N/m$. Then the force at any distance will be $F(x) = 90x$, so the work is

$$W = \int_0^5 F(x)dx = \int_0^5 90xdx = 45x^2\Big|_0^5 = 1125.$$

6) Again, $k = F/x = (150lb)/(1/16in) = 16\cdot150lb/in$, so for the first part $F(1/8) = (16\cdot150lb/in)(1/8in) = 300lb$, and the second part is

$$W(1/8) = \int_0^{1/8} (16 \cdot 150)xdx = (8 \cdot 150)x^2\Big|_0^{1/8} = \frac{150}{8} = \frac{75}{4}ft - lb$$

For rope and tank problems we pretend to lift a little piece at a time and integrate over the boundary.

7) The density is $\rho = 0.624$ N/m, so the weight of some arbitrary i^{th} piece is $F_i = (0.624)\delta x$. So the work for that piece is $W_i = F_i x_i^* = [(0.624)\delta x]x_i^*$. If we can do this for one piece, we can do it for n pieces, so $W \approx \sum_{i=1}^n x_i^* \delta x$. Taking the limit gives us

$$W = \lim_{n\to\infty}\sum_{i=1}^n x_i^*\delta x = \int_0^{50} (0.624)xdx = (0.312)x^2\Big|_0^{50} = (0.312)50^2 = 780J$$

9) Again, $\rho = 4.5lb/ft \Rightarrow F_i = 4.5\delta x \Rightarrow W_i = (4.5\delta x)x_i^*$, then

$$W = \int_0^{180} 4.5xdx = \frac{9}{4}x^2\Big|_0^{180} = 9 \cdot 90^2 = 72,900ft - lb$$

15) Lets define our coordinate system to be 0 at the top and 10 at the bottom. We break the tank up into circular cylinders, such that the i^{th} cylinder has a height of δx_i and a radius of r_i. We need to find r_i in terms of x_i. We can do this by using similar triangles, i.e. the ratio of the radii will be equivalent to the ratio of the heights of the big triangle (half the cross-section of the tank) and the small triangle (half the cross-section of the water). The ratio is

$$\frac{r_i}{5} = \frac{10 - x_i^*}{10} \Rightarrow r_i = \frac{1}{2}(10 - x_i^*).$$

Now we know the volume, force, and work of the i^{th} cylinder

$$V_i = \pi r_i^2 \delta x_i = \frac{\pi}{4}(10 - x_i^*)^2 \delta x_i \Rightarrow F_i = \rho V_i = \frac{57}{4}\pi(10 - x_i^*)^2 \Delta x_i \Rightarrow W_i = F_i x_i^* = \frac{57}{4}\pi x_i^*(10 - x_i^*)\Delta x_i$$

Then the work is

$$W = \frac{57}{4}\pi \int_0^{10} x(10-x)^2 dx = \frac{57}{4}\pi \int_0^{10} (100x-20x^2+x^3)dx = \frac{57}{4}\pi \left[50x^2 - \frac{20}{3}x^3 + \frac{1}{4}x^4\right]_0^{10} = \frac{57}{4}\pi[5000-20000/3+2500]$$

16) The only thing that changes are the limits and the distance traveled, so

$$W_i = F_i \cdot (4 + x_i) \Rightarrow W = \frac{57}{4}\pi \int_5^{10} (4 + x)(100 - 20x + x^2)dx$$

17) This is a much easier problem because the radius doesn't change, so

$$V_i = \pi r_i^2 h_i = \pi 10^2 \Delta x_i = 100\pi \Delta x_i \Rightarrow F_i = \rho v_i = 5120\pi \Delta x_i \Rightarrow W_i = F_i x_i = 5120\pi x_i \Delta x_i$$

Then the work is

$$W = 5120\pi \int_0^{30} x dx = 5120\pi \cdot \frac{1}{2}x^2 \Big|_0^{30} = 2560\pi \cdot 30^2 ft - lb.$$

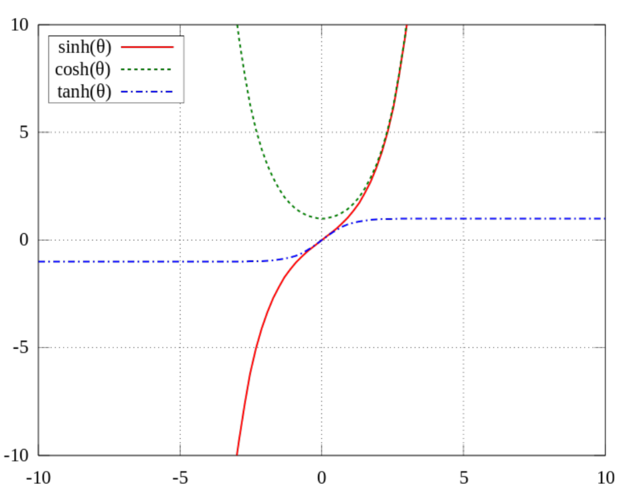
1. 7.3 Hyperbolic Functions

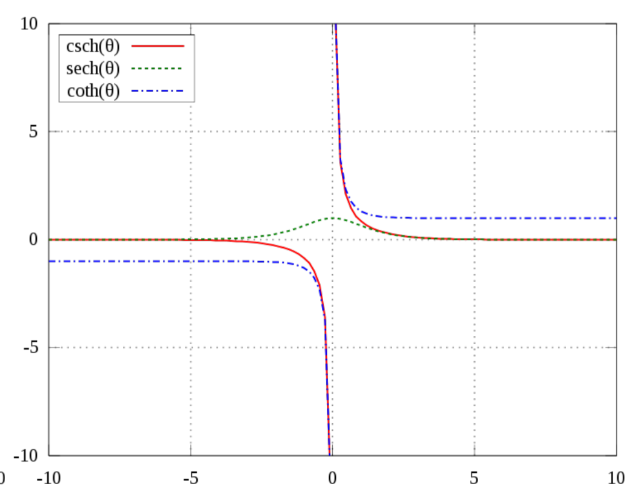
Hyperbolic functions are similar to trigonometric functions, and have the following definitions:

$\sinh x = \frac{1}{2}(e^x - e^{-x})$ $\qquad \cosh x = \frac{1}{2}(e^x + e^{-x})$ $\qquad \tanh x = \frac{\sinh x}{\cosh x}$

$\operatorname{csch} x = 1/\sinh x$ $\qquad \operatorname{sech} x = 1/\cosh x$ $\qquad \coth x = 1/\tanh x$

It is also useful to know what they look like. You don't have to be able to graph them precisely, just have an idea of the sketch. In order to recall what they look like just use the definitions and take the average of the exponentials.





They are also subject to the following identities:

$\sinh(-x) = -\sinh(x)$ $\cos(-x) = \cosh x$ $\cosh^2 x - \sinh^2 x = 1$ $1 - \tanh^2 x = \operatorname{sech}^2 x$

$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$ $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

Proving these identities will allow us to understand them better.

Theorem 1. $\cosh^2 x - \sinh^2 x = 1$

Proof.

$$\cosh^2 x - \sinh^2 x = \left[\frac{1}{2}(e^x + e^{-x})\right]^2 - \left[\frac{1}{2}(e^x - e^{-x})\right]^2 = \frac{1}{4}(e^{2x} + 2 + e^{-2x}) - \frac{1}{4}(e^{2x} - 2 + e^{-2x}) = 1$$

□

Theorem 2. $1 - \tanh^2 x = \operatorname{sech}^2 x$

Proof. Here we simply divide the entire equation by $\cosh^2 x$,

$$[\cosh^2 x - \sinh^2 x = 1]\frac{1}{\cosh^2 x} \Rightarrow 1 - \tanh^2 x = \operatorname{sech}^2 x$$

□

The other identities are proved in a similar manner. Even though proofs don't appear on exams they will help you get a better understanding of the concepts.

It's also important to know the derivatives of hyperbolic functions,

$(\sinh x)' = \cosh x$ $(\cosh x)' = \sinh x$ $(\tanh x)' = \operatorname{sech}^2 x$

$(\coth x)' = -\operatorname{csch}^2 x$ $(\operatorname{sech} x)' = -\operatorname{sech} x \tanh x$ $(\operatorname{csch} x)' = -\operatorname{csch} x \coth x$

These can all be derived very easily straight from the definitions. Lets do one example derivative

$$(\cosh\sqrt{x})' = \frac{1}{2\sqrt{x}}(\sinh\sqrt{x})$$

8.1 Integration Review

Additional problems that were not done in class:

4) We first simplify the integral,

$$I = \int_{\pi/4}^{\pi/3} \frac{dx}{\cos^2 x \tan x} = \int_{\pi/3}^{\pi/4} \sec x dx = \int_{\pi/3}^{\pi/4} \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx = \int_{\pi/3}^{\pi/4} \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} x dx$$

Then we can do a u-sub, $u = \sec x + \tan x \Rightarrow du = (\sec^2 x + \sec x \tan x)dx$, so

$$I = \int_{\pi/4}^{\pi/3} \frac{du}{u} = \ln|u|\Big|_{\pi/4}^{\pi/3} = \ln|\sec x + \tan x|\Big|_{\pi/4}^{\pi/3} = \ln|2+\sqrt{3}| - \ln|1+\sqrt{2}|.$$

18) Let $u = \sqrt{y} \Rightarrow du = 1/2\sqrt{y}$, so

$$I = \int 2^u du = \int e^{u \ln 2} = \frac{1}{\ln 2} e^{u \ln 2} = \frac{1}{\ln 2} 2^u = \frac{1}{\ln 2} 2^{\sqrt{y}}$$

40) Let $u = x^{3/2} \Rightarrow du = \frac{3}{2}\sqrt{x}dx$, then

$$I = \frac{2}{3}\int \frac{du}{1+u^2} = \frac{2}{3}\tan^{-1} u = \frac{2}{3}\tan^{-1}(x^{3/2}) + C$$

8.2 Integration by Parts

The modern notion of integration by parts comes from an extensive theory by Riemann and Stieltjes in 1894, soon after which Stieltjes passed away. The idea is we can integrate over certain functions instead of just over x. We can think of it as a generalization of "u-sub".

To derive it, we consider the product rule,

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x) \Rightarrow d[f(x)g(x)] = f(x)g'(x)dx + g(x)f'(x)dx$$

$$\Rightarrow \int d[f(x)g(x)] = f(x)g(x) = \int f(x)g'(x)dx + \int g(x)f'(x)dx \Rightarrow \int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

This can be written in shorthand as

$$\int udv = uv - \int vdu \tag{1}$$

As discussed in class we generally choose the easiest thing to integrate as dv, and the other as u. We can use ILATE: InverseLogsAlgebraicTrigonometricExponential, to help determine which is easier to integrate. Things get easier to integrate as we go to the right, for example, Exponentials are easier to integrate than Trigonometric functions. But this doesn't always work! So, only use it as a guide, not even as a rule of thumb. Also, try to avoid using "tabular", it doesn't save that much time and it increases the chance of a mistake.

Here are some problems we did in class

5) Since polynomials are easier to integrate than logs we use $u = \ln x \Rightarrow du = dx/x$ and $dv = xdx \Rightarrow v = x^2/2$, then

$$I = \frac{1}{2}x^2 \ln x\Big|_1^2 - \frac{1}{2}\int_1^2 xdx = 2\ln 2 - \frac{1}{4}x^2\Big|_1^2 = 2\ln 2 - [1 - \frac{1}{4}] = 2\ln 2 - \frac{3}{4}.$$

11) Let $u = \tan^{-1} y \Rightarrow du = dy/(1+y^2)$ and $dv = dy \Rightarrow v = y$, so

$$I = y\tan^{-1} y - \int \frac{ydy}{1+y^2}$$

Now we use u-sub, $u = 1 + y^2 \Rightarrow du = 2ydy$, then

$$I = y\tan^{-1} y - \frac{1}{2}\int \frac{du}{u} = y\tan^{-1} y - \frac{1}{2}\ln|u| = y\tan^{-1} y - \frac{1}{2}\ln(1+y^2) + C.$$

27) We first convert this to

$$I = \int_0^{\pi/3} x(\sec^2 x - 1)dx = -\frac{1}{2}x^2\Big|_0^{\pi/3} + \int_0^{\pi/3} x\sec^2 xdx$$

Now, we employ by parts, $u = x \Rightarrow du = dx$ and $dv = \sec^2 xdx \Rightarrow v = \tan x$,

$$I = -\frac{\pi^2}{18} + x\tan x\Big|_0^{\pi/3} - \int_0^{\pi/3} \tan xdx = -\frac{\pi^2}{18} + \frac{\pi}{3}\sqrt{3} - \int_0^{\pi/3} \frac{\sin x}{\cos x}dx$$

Then we use u-sub, $u = \cos x \Rightarrow du = -\sin xdx$,

$$I = -\frac{\pi^2}{18} + \frac{\pi}{3}\sqrt{3} - \int_1^{1/2} \frac{du}{u} = -\frac{\pi^2}{18} + \frac{\pi}{3}\sqrt{3} + \ln u\Big|_1^{1/2}$$

33) Let $u = (\ln x)^2 \Rightarrow 2(\ln x)/x$ and $dv = xdx \Rightarrow v = x^2/2$, then

$$I = \frac{1}{2}x^2(\ln x)^2 - \int x\ln xdx$$

This was already solved in problem 5, so

$$I = \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2 + C$$

37) Here we don't need by parts, we can just use u-sub, $u = x^4 \Rightarrow du = 4x^3$, then

$$I = \frac{1}{4}\int e^u du = \frac{1}{4}e^u + C = \frac{1}{4}e^{x^4} + C.$$

39) We use only u-sub again, $u = x^2 + 1 \Rightarrow du = 2xdx$, then

$$I = \frac{1}{2}\int (u-1)\sqrt{u}du = \frac{1}{2}\int (u^{3/2} - u^{1/2})du = \frac{1}{2}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right] + C = \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C.$$

29) First we do a u-sub, $\theta = \ln x \Rightarrow d\theta = dx/x$, so $x = e^\theta \Rightarrow dx = e^\theta d\theta$. Therefore our integral becomes $I = \int e^\theta \sin\theta d\theta$. Now we can use by parts, $u = \sin\theta \Rightarrow du = \cos\theta d\theta$ and $dv = e^\theta d\theta \Rightarrow v = e^\theta$,

$$I = e^\theta \sin\theta - \int e^\theta \cos\theta d\theta$$

Now we use by parts again, but remember try to avoid tabular, $u = \cos\theta \Rightarrow du = -\sin\theta d\theta$ and $dv = e^\theta d\theta \Rightarrow v = e^\theta$, so

$$I = e^\theta \sin\theta - \left[e^\theta \cos\theta + \int e^\theta \sin\theta d\theta\right] = e^\theta \sin\theta - \left[e^\theta \cos\theta + I\right]$$

We notice that by doing the second by parts we get our original integral back, so now we can do some algebra

$$2I = e^\theta \sin\theta - e^\theta \cos\theta \Rightarrow I = \frac{1}{2}\left[e^\theta \sin\theta - e^\theta \cos\theta\right] = \frac{1}{2}\left[x\sin(\ln x) - x\cos(\ln x)\right].$$

53) $I = \int_a^b x\sin x dx = -x\cos x\big|_a^b + \int \cos x dx = -x\cos x + \sin x\big|_a^b$

(a) $I = -x\cos x + \sin x\big|_0^\pi = \pi \Rightarrow A = \pi$

(b) $I = -x\cos x + \sin x\big|_\pi^{2\pi} = -2\pi - \pi \Rightarrow A = 3\pi$

(c) $I = -x\cos x + \sin x\big|_{2\pi}^{3\pi} = 3\pi + 2\pi \Rightarrow A = 5\pi$

(d) $A = (2n+1)\pi$; $n \in \mathbb{Z}^+$; i.e. $n = 0, 1, 2, \ldots$

Additional problems (not in the book)

Ex: $I = \int \ln x dx$

Solution: Let $u = \ln x \Rightarrow du = dx/x$ and $dv = dx \Rightarrow v = x$. Then

$$I = x\ln x - \int x\frac{dx}{x} = x\ln x - x + C$$

Ex: $I = \int t^2 e^t dt$.

Solution: Let $u = t^2 \Rightarrow du = 2tdt$ and $dv = e^t dt \Rightarrow v = e^t$. Then

$$I = t^2 e^t - 2\int te^t dt$$

Notice, we need to use by parts again. Let $u = t \Rightarrow du = dt$ and $dv = e^t dt \Rightarrow v = e^t$. Then

$$I = t^2 e^t - 2\left[te^t - \int e^t dt\right] = t^2 e^t - 2te^t + 2e^t + C.$$

It may be appealing to do this sort of problem using “tabular integration”, however you should avoid using this shortcut. If you happen to use it and get it wrong on the exam you will end up losing more points than if you just did by parts twice, and it doesn’t save you that much time.

Ex $I = \int \sin^n x dx$.

Solution: This type of example won't show up on the exam. This is purely theoretical so you can ignore it, but if you are interested it's a good problem to test your abstraction abilities.

Let $u = \sin^{n-1} x \Rightarrow du = (n-1)\sin^{n-2} x \cos x dx$ and $dv = \sin x dx \Rightarrow v = -\cos x$. Then

$$\begin{aligned}
\int sin^n x dx &= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \cos^2 x dx \\
&= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)dx \\
&= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x - (n-1)\int \sin^n x dx \\
&\Rightarrow n \sin^n x dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx \\
&\Rightarrow \sin^n x dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x dx
\end{aligned}$$

8.3 Trigonometric Integrals

Lets first look at a few examples

Sines and Cosines.

9) Here we can replace the $\cos^2 x$ by $1 - \sin^2 x$ and have a $\cos x$ left over

$$I = \int \cos^3 x dx = \int (1-\sin^2 x)\cos x dx = \int \cos x dx - \int \sin^2 x \cos x dx = \sin x - \int \sin^2 x \cos x dx$$

Then we can use u-sub, $u = \sin x \Rightarrow du = \cos x dx$,

$$I = \sin x - \int u^2 du = \sin x - \frac{1}{3}u^3 + C = \sin x - \frac{1}{3}\sin^3 x + C$$

11) For this problem we can either substitute for $\cos^2 x$ or $\sin^2 x$, lets go with sine because it produces a positive derivative for the u-sub,

$$I = \int \sin^3 x(1-\sin^2 x)\cos x dx = \int (\sin^3 x - \sin^5 x)\cos x dx$$

Then we have $u = \sin x \Rightarrow du = \cos x dx$, then

$$I = \int (u^3 - u^5)du = \frac{1}{4}u^4 - \frac{1}{6}u^6 + C = \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$$

17) Here we don't have a mix of sine and cosine, so we need to use a different identity. How about $\sin^2 x = \frac{1}{2}(1-\cos 2x)$,

$$I = 8\int_0^\pi \left[\frac{1}{2}(1-\cos 2x)\right]^2 dx = 2\int_0^\pi (1 - 2\cos 2x + \cos^2 x)dx = 2\int_0^\pi \left[1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x)\right] dx$$

$$= 2\left[\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right]_0^\pi = 3\pi$$

Strategies for $\int \sin^m x \cos^n x dx$

(1) If the power of the cosine term is odd (i.e. $n = 2k+1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$,

$$\int \sin^m x \cos^{2k+1} x dx = \int \sin^m x (\cos^2 x)^k \cos x dx = \int \sin^m x (1-\sin^2 x)^k \cos x dx \quad (2)$$

Then substitute $u = \sin x \Rightarrow du = \cos x dx$.

(2) If the power of the sine term is odd (i.e. $m = 2k+1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$,

$$\int \sin^{2k+1} x \cos^n x dx = \int (\sin^2 x)^k \cos^n x dx = \int (1-\cos^2 x)^k \cos^n x \sin x dx \quad (3)$$

Then substitute $u = \cos x \Rightarrow du = -\sin x dx$

(3) If the powers of both sine and cosine are even, use the double-angle formulas:

$$\sin^2 x = \frac{1}{2}(1-\cos 2x) \qquad \cos^2 x = \frac{1}{2}(1+\cos 2x) \qquad \sin x \cos x = \frac{1}{2}\sin 2x$$

Tangents and Secants.

35) **Method1:** We can convert this into sines and cosines and use u-sub, $u = \cos x \Rightarrow du = -\sin x dx$

$$I = \int \frac{\sin x}{\cos^4 x} dx = -\int \frac{du}{u^4} = \frac{1}{3}u^{-3} + C = \frac{1}{3}\sec^3 x + C$$

Method2: We can also separate out a $\sec x \tan x$ and do the u-sub $u = \sec x \Rightarrow du = \sec x \tan x dx$

$$I = \int \sec^2 x (\sec x \tan x) dx = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}\sec^3 x + C$$

37) Here we can substitute $u = \tan x \Rightarrow du = \sec^2 x dx$,

$$I = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3 x + C$$

Strategy for $\int \tan^m x \sec^n x dx$

(1) If the power of the secant term is even (i.e. $n = 2k$, $k \geq 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$,

$$\int \tan^m x \sec^{2k} x dx = \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x dx = \int \tan^m x (1+\tan^2 x)^{k-1} \sec^2 x dx \quad (4)$$

Then substitute $u = \tan x \Rightarrow du = \sec^2 x dx$

(2) If the power of the tangent term is odd (i.e. $m = 2k+1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$,

$$\int \tan^{2k+1} x \sec^n x dx = \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x dx = \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x dx$$

Then substitute $u = \sec x \Rightarrow du = \sec x \tan x dx$

Useful Integrals

These integrals are pretty easy to derive if you forget them,

$$\int \tan x dx = -\ln|\cos x| + C = \ln|\sec x| + C \quad (5)$$

$$\int \sec x dx = \ln|\sec x + \tan x| + C. \quad (6)$$

Here are some more problems we did in class

38) Here we use $u = \tan x \Rightarrow du = \sec^2 x dx$,

$$I = \int (1 + \tan^2 x) \tan^2 x \sec^2 x dx = \int (u^2 + u^4) du = \frac{1}{3}u^3 + \frac{1}{5}u^5 + C = \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C$$

19) Here we use our double angle formulas

$$I = 16 \int \left[\frac{1}{2}(1 - \cos 2x)\right]\left[\frac{1}{2}(1 + \cos 2x)\right] dx = 4 \int (1 - \cos^2 2x) dx = 4x - 4 \int \frac{1}{2}(1 + \cos 4x) dx$$

$$= 4x - 2x - \frac{1}{2}\sin 4x + C = 2x - \frac{1}{2}\sin 4x + C.$$

65) We use by parts $u = \tan^2 x \Rightarrow du = 2 \tan x \sec^2 x dx$ and $dv = \sin x dx \Rightarrow v = -\cos x$

$$I = \int \sin x \tan^2 x dx = -\cos x \tan^2 x - 2 \int \cancel{\cos x} \frac{\sin x}{\cancel{\cos x}} \frac{1}{\cos^2 x} dx = -\cos x \tan^2 x - 2 \int \frac{\sin x}{\cos^2 x} dx$$

Then we use u-sub, $u = \cos x \Rightarrow du = -\sin x dx$,

$$I = -\cos x \tan^2 x + 2 \int \frac{du}{u^2} = -\cos x \tan^2 x - 2u^{-1} + C = -\cos x \tan^2 x - 2 \sec x + C.$$

67) Here we use our double angle formula

$$I = \int x \left[\frac{1}{2}(1 - \cos 2x)\right] dx = \int \left(\frac{x}{2} - x \cos 2x\right) dx = \frac{x^2}{4} - \int x \cos 2x dx$$

Now we use by parts, $u = x \Rightarrow du = dx$ and $dv = \cos 2x dx \Rightarrow v = \frac{1}{2}\sin 2x$

$$I = \frac{x^2}{4} - \frac{x}{2}\sin 2x + \frac{1}{2}\int \sin 2x dx = \frac{x^2}{4} - \frac{x}{2}\sin 2x - \frac{1}{4}\cos 2x + C$$

Additional problems (not in the book)

Ex: $I = \int \sin^5 x \cos^2 x dx$

Solution: We substitute for sine until there is only one left,

$$I = \int (\sin^2 x)^2 \cos^2 x \sin x dx = \int (1 - \cos^2 x)^2 \cos^2 x \sin x dx$$

Then we do u-sub, $u = \cos x \Rightarrow du = -\sin x dx$,

$$\int \sin^5 x \cos^2 x dx = -\int (1 - u^2)^2 u^2 du = -\frac{1}{3}u^3 + \frac{2}{5}u^5 - \frac{1}{7}u^7 + C = -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$$

Ex: $I = \int_0^\pi \sin^2 x dx$

Solution: Again, we use the double angle identity,

$$I = \int_0^\pi \sin^2 x dx = \frac{1}{2}\int_0^\pi (1 - \cos 2x) dx = \left[\frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)\right]_0^\pi = \frac{\pi}{2}$$

Ex: $I = \int \tan^6 x \sec^4 x dx$

Solution: We substitute in $\sec^2 x = 1 + \tan^2 x$ until a single $\sec^2 x$ is left. Then we substitute $u = \tan x \Rightarrow du = \sec^2 x dx$

$$I = \int \tan^6 x (1 + \tan^2 x) \sec^2 x dx = \int u^6 (1 + u^2) du = \frac{1}{7}u^7 + \frac{1}{9}u^9 + C = \frac{1}{7}\tan^7 x + \frac{1}{9}\tan^9 x + C$$

Ex: $\int \tan^5 \theta \sec^7 \theta d\theta$.

Solution: Here we substitute $\tan^2 x = \sec^2 x - 1$ until a single $\sec x \tan x$ remains

$$I = \int \tan^4 \theta \sec^6 \theta (\sec\theta \tan\theta) d\theta = \int (\sec^2 \theta - 1)^2 \sec^6 \theta (\sec\theta \tan\theta) d\theta$$

Then we use u-sub, $u = \sec\theta \Rightarrow du = (\sec\theta \tan\theta) d\theta$

$$I = \int (u^2 - 1)^2 u^6 du = \frac{1}{11}u^{11} - \frac{2}{9}u^9 + \frac{1}{7}u^7 + C = \frac{1}{11}\sec^{11} x - \frac{2}{9}\sec^9 x + \frac{1}{7}\sec^7 x + C$$

Ex: $I = \int \tan^3 x dx$

Solution: We use the identity $\tan^2 x = \sec^2 x - 1$, and $u = \tan x \Rightarrow du = \sec^2 x dx$

$$I = \int \tan x(\sec^2 x - 1)dx = \frac{1}{2}\tan^2 x + \ln|\cos x| + C$$

Ex: $I = \sec^3 x dx$

Solution: We integrate by parts with $u = \sec x \Rightarrow du = \sec x \tan x dx$ and $dv = \sec^2 x \Rightarrow v = \tan x$, then

$$I = \sec x \tan x - \int \sec x \tan^2 x dx = \sec x \tan x - \int \sec x(\sec^2 x - 1)dx = \sec x \tan x - \int \sec^3 x dx + \int \sec x dx$$

$$= \sec x \tan x - \int \sec^3 x dx + \ln|\sec x + \tan x| = \sec x \tan x + \ln|\sec x + \tan x| - I$$

$$\Rightarrow \int \sec^3 x dx = \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x|\right] + C$$