## 8.4 Trigonometric Substitution

Let's begin with an example,

7) If we pull out a 5 the expression looks a lot like $1 - \sin^2\theta = \cos^2\theta$, so lets use the substitution $t = 5\sin\theta \Rightarrow dt = 5\cos\theta d\theta$,

$$I = \int \left(\sqrt{25 - 25\sin^2\theta}\right) 5\cos\theta d\theta = 25\int \left(\sqrt{1-\sin^2\theta}\right)\cos\theta d\theta = 25\int\left(\sqrt{\cos^2\theta}\right)\cos\theta d\theta = 25\int \cos^2\theta d\theta$$
$$= \frac{25}{2}\int(1+\cos 2\theta)d\theta = \frac{25}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C = \frac{25}{2}\left[\theta + \sin\theta\cos\theta\right] + C = \frac{25}{2}\sin^{-1}\frac{t}{5} + \frac{t}{2}\sqrt{25-t^2} + C.$$

The tricky part is going from $\theta$ to $x$. We see that since $x = 5\sin\theta$, $\cos\theta = \frac{1}{5}\sqrt{25 - 25\sin^2\theta} = \frac{1}{5}\sqrt{25-t^2}$. We should also use right triangles to help us out.

Let us employ a table of trigonometric substitutions that will help us with the decision making process

| Expression | Substitution | Identity |
|---|---|---|
| $a^2 - x^2$ | $x = a\sin\theta,\ -\pi/2 \le \theta \le \pi/2$ | $1 - \sin^2\theta = \cos^2\theta$ |
| $a^2 + x^2$ | $x = a\tan\theta,\ -\pi/2 < \theta < \pi/2$ | $1 + \tan^2\theta = \sec^2\theta$ |
| $x^2 - a^2$ | $x = a\sec\theta,\ 0 \le \theta < \pi/2,\ \pi \le \theta < 3\pi/2$ | $\sec^2\theta = 1 - \tan^2\theta$ |

Here are some problems we did in class

29) For this we use $x = \frac{1}{2}\tan\theta \Rightarrow dx = \frac{1}{2}\sec^2\theta d\theta$. Then we get

$$I = \int\frac{4\sec^2\theta d\theta}{(\sec^2\theta)^2} = 4\int\cos^2\theta d\theta = 2\int(1+\cos 2\theta)d\theta = 2\left[\theta + \frac{1}{2}\sin 2\theta\right] + C = 2[\theta + \sin\theta\cos\theta] + C = 2\tan^{-1}(2x) + \frac{4x}{1+4x^2} + C.$$

35) First we do a u-sub $u = e^t \Rightarrow du = e^t dt$ to get $I = \int_1^4 du/\sqrt{u^2+9}$. Then we use $u = 3\tan\theta \Rightarrow du = 3\sec^2\theta d\theta$,

$$I = \int_1^4 \frac{\sec^2\theta}{\sec\theta}d\theta = \ln|\tan\theta + \sec\theta| = \ln\left|\frac{u}{3} + \frac{1}{3}\sqrt{9+u^2}\right|\Bigg|_1^4 = \ln\frac{9}{1+\sqrt{10}}.$$

39) Here we use $x = \sec\theta \Rightarrow dx = \sec\theta\tan\theta d\theta$,

$$I = \int\frac{\sec\theta\tan\theta d\theta}{\sec\theta\sqrt{\sec^2\theta - 1}} = \int\frac{\tan\theta}{\tan\theta}d\theta = \sec^{-1}x + C.$$

43) Here we do something slightly different, $x^2 = \tan\theta \Rightarrow 2xdx = \sec^2\theta d\theta$,

$$I = \frac{1}{2}\int\frac{\sec^2\theta d\theta}{\sqrt{1+\tan^2\theta}} = \frac{1}{2}\int\frac{\sec^2\theta}{\sec\theta} = \frac{1}{2}\ln|\tan\theta + \sec\theta| + C = \frac{1}{2}\ln|x^2 + \sqrt{1+x^2}| + C.$$

Here are some additional problems that are not in the book,

Ex: $I = \int \frac{\sqrt{9-x^2}}{x^2}dx$.

**Solution:** This is of the form of the first case, so we use $x = 3\sin\theta \Rightarrow dx = 3\cos\theta d\theta$,

$$I = \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}3\cos\theta d\theta = \int \frac{9\cos^2\theta}{9\sin^2\theta}d\theta = \int \cot^2\theta d\theta = \int(\csc^2\theta - 1)d\theta = -\cot\theta - \theta + C.$$

Now we must plug back in for $\theta$. Since $x = 3\sin\theta$, $\sin\theta = x/3$, and we recall that sine is opposite over hypotenuse and cotangent is adjacent over opposite. We may denote the opposite side as $x$ and the hypotenuse as 3, then by the Pythagorean theorem the adjacent side is $\sqrt{9-x^2}$. This gives us, $\cot\theta = \sqrt{9-x^2}/x$, then

$$I = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C.$$

Ex: $I = \int \frac{dx}{x^2\sqrt{x^2+4}}$.

**Solution:** This is of the form of the second case, so we substitute $x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta d\theta$,

$$I = \int \frac{2\sec^2\theta d\theta}{4\tan^2\theta\sqrt{4\tan^2\theta+4}} = \frac{1}{4}\int\frac{\sec\theta}{\tan^2\theta}d\theta = \frac{1}{4}\int\frac{\cos\theta}{\sin^2\theta}d\theta.$$

We solve this via u-sub with $u = \sin\theta \Rightarrow du = \cos\theta d\theta$,

$$I = \frac{1}{4}\int\frac{du}{u^2} = -\frac{1}{4u} + C = -\frac{1}{\sin\theta} + C.$$

Since $x = 2\tan\theta$, $\sin\theta = x/\sqrt{x^2+4}$, then

$$I = -\frac{\sqrt{x^2+4}}{4x} + C.$$

Ex: $I = \int \frac{xdx}{\sqrt{x^2+4}}$.

**Solution:** Thought we had to use a trig substitution didn't ya? NOPE! U-Sub! Let $u = x^2 + 4 \Rightarrow du = 2xdx$,

$$I = \frac{1}{2}\int\frac{du}{\sqrt{u}} = \sqrt{u} + C = \sqrt{x^2+4} + C.$$

This problem shows us that if we take a few seconds to think about a problem we can find a much easier solution.

Ex: $\int \frac{dx}{\sqrt{x^2-a^2}}$.

**Solution:** This is of the form of the third case, so we substitute $x = a\sec\theta \Rightarrow dx = a\sec\theta\tan\theta$.

$$I = \int\frac{a\sec\theta\tan\theta d\theta}{\sqrt{a^2\sec^2\theta - a^2}} = \int\frac{a\sec\theta\tan\theta d\theta}{a\tan\theta} = \int\sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C$$

Since $x = a\sec\theta$, $\tan\theta = \sqrt{x^2-a^2}/a$, then

$$I = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C$$

Notice $a\cosh x$ is an equivalent way to do the problem, but not the preferred method for the exams.

Ex: $\int_0^{3\sqrt{3}/2} \frac{x^3 dx}{(4x^2+9)^{3/2}}$.

**Solution:** This is of the form of the second case, but here we have a coefficient in front of the $x$ term. We can either pull the 4 out and then start our calculations or we can see what $x$ has to be with the 4 there. It's much easier to come up with a substitution for $x$ that produces a desired result than to pull the coefficient out. Notice that we need the coefficient in front of the $\tan^2$ term (after substitution of course) to be 9, so we have that $4x^2 = 9\tan^2\theta$, then $x = (3/2)\tan\theta \Rightarrow dx = (3/2)\sec^2\theta d\theta$,

$$I = \left(\frac{3}{2}\right)^4 \int_0^{\pi/3} \frac{\tan^3\theta \sec^2\theta d\theta}{(9\tan^2\theta + 9)^{3/2}} = \left(\frac{3}{2}\right)^4 \int_0^{\pi/3} \frac{\tan^3\theta \sec^2\theta d\theta}{3^3\sec^3\theta} = \frac{3}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta} d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} \frac{\sin^3\theta}{\cos^2\theta} d\theta = \frac{3}{16}\int_0^{\pi/3} \frac{1-\cos^2\theta}{\cos\theta}\sin\theta d\theta$$

This is our usual trig integral where $u = \cos\theta \Rightarrow du = -\sin\theta d\theta$,

$$I = -\frac{3}{16}\int_1^{1/2} \frac{1-u^2}{u^2} du = \frac{3}{16}\left[u + \frac{1}{u}\right]_1^{1/2} = \frac{3}{32}.$$

Ex: $I = \int \frac{x dx}{\sqrt{3-2x-x^2}}$.

**Solution:** This one is going to take a bit of ingenuity. Lets tinker with $3-2x-x^2 = 3-(x^2+2x)$. Notice we can get a perfect square if we add a 1 to $x^2+2x$, but if we add a 1 we must also subtract a 1, so $3-2x-x^2 = 3-(x^2+2x+1)+1 = 4-(x+1)^2$. Now, let $u = x+1 \Rightarrow du = dx$, then

$$I = \int \frac{(u-1)du}{\sqrt{4-u^2}}.$$

This is precisely the form of the first case, so we substitute $u = 2\sin\theta \Rightarrow du = 2\cos\theta d\theta$

$$I = \int \frac{2\sin\theta - 1}{\sqrt{4-4\sin^2\theta}} 2\cos\theta d\theta = \int \frac{2\sin\theta - 1}{2\cos\theta} 2\cos\theta d\theta = -2\cos\theta - \theta + C$$

Since $u = 2\sin\theta$, $2\cos\theta = \sqrt{4-u^2}$, so

$$I = -\sqrt{4-u^2} - \sin^{-1}\frac{u}{2} + C = \sqrt{3-2x-x^2} - \sin^{-1}\frac{x+1}{2} + C.$$

## 8.5 Integration of Rational Functions by Partial Fractions

This is an algebraic trick, not calculus. We can see this in two simple examples

$$\text{Ex: } \frac{2x}{x^2-1} = \frac{1}{x-1} + \frac{1}{x+1} \qquad \text{Ex: } \frac{2}{x^2-1} = \frac{1}{x-1} - \frac{1}{x+1}$$

From this point we will consider integrals of the type

$$\int f(x)dx; \qquad f(x) = \frac{P(x)}{Q(x)}, \text{ where } P \text{ and } Q \text{ are polynomials.} \tag{1}$$

We can think of four different cases that we can split into partial fractions.

**Case 1**

Suppose $Q$ is a product of distinct linear factors, i.e. $Q = (a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$. Then,

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}. \tag{2}$$

Here is an example of the first case

11) We can split the fraction as follows

$$\frac{x+4}{(x+6)(x-1)} = \frac{A}{x+6} + \frac{B}{x-1} = \frac{Ax - A + Bx + 6B}{(x+6)(x-1)} = \frac{x+4}{(x+6)(x-1)} \Rightarrow (A+B)x + (6B - A) = x + 4$$

Matching terms gives us $A + B = 1$ and $-A + 6B = 4$, then $7B = 5 \Rightarrow B = 5/7 \Rightarrow A = 2/7$. Then the integral becomes

$$I = \frac{2}{7}\int \frac{dx}{x+6} + \frac{5}{7}\int \frac{dx}{x-1} = \frac{2}{7}\ln|x+6| + \frac{5}{7}\ln|x-1| + C.$$

Now lets look nondistinct linear factors, for example

3)

$$\frac{x+4}{(x+1)^2} = \frac{x+1}{(x+1)^2} + \frac{3}{(x+1)^2} = \frac{1}{x+1} + \frac{3}{(x+1)^2}.$$

We notice that we can't fully separate this.

**Case 2**

Suppose $Q$ is a product of linear factors, some of which are repeated. Then the repeated factors are of the form

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(ax+b)^r} = \frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_r}{(ax+b)^r}. \tag{3}$$

Here is an example of Case 2.

19) We can split the fractions as follows

$$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$
$$\Rightarrow A(x-1)(x+1)^2 + B(x+1)^2 + C(x+1)(x-1)^2 + D(x-1)^2$$
$$= (A+C)x^3 + (A+B-C+D)x^2 + (-A+2B-C-2D)x + (-A+B+C+D) = 1$$

Matching terms gives us $A + C = 0 \Rightarrow A = -C$, so $B = D \Rightarrow B = C$, then $4C = 1 \Rightarrow C = 1/4 = B = D = -A$. Then our integral becomes

$$I = -\frac{1}{4}\int \frac{dx}{x-1} + \frac{1}{4}\int \frac{dx}{(x-1)^2} + \frac{1}{4}\int \frac{dx}{x+1} + \frac{1}{4}\frac{dx}{(x+1)^2} = -\frac{1}{4}\ln|x-1| - \frac{1}{4}(x-1)^{-1} + \frac{1}{4}\ln|x+1| - \frac{1}{4}(x+1)^{-1} + C$$

We can also have quadratic factors

**Case 3**

Suppose $Q$ is a product of quadratic factors with no repeats; i.e. $Q = (a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x + c_2)\cdots(a_kx^2 + b_kx + c_k)$, then

$$\begin{aligned}\frac{P(x)}{Q(x)} &= \frac{P(x)}{(a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x + c_2)\cdots(a_kx^2 + b_kx + c_k)} \\ &= \frac{A_1x + B_1}{a_1x^2 + b_1x + c_1} + \frac{A_2x + B_2}{a_2x^2 + b_2x + c_2} + \cdots + \frac{A_kx + B_k}{a_kx^2 + b_kx + c_k}.\end{aligned} \tag{4}$$

Lets look at the following example,

Ex:

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{x^2+4}$$
$$\Rightarrow A(x^2+1)(x^2+4) + (Bx + C)(x-2)(x^2+4) + (Dx + F)(x-2)(x^2+4) = x.$$

Here is an example of Case 3,

27) After factoring we get $(x^2-x+2)/(x-1)(x^2+x+1)$. Then we can split this up

$$\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1} \Rightarrow A(x^2+x+1)+(Bx+C)(x-1) = A(x^2+x+1)+(Bx^2-Bx+Cx-C)$$
$$= (A+B)x^2+(A-B+C)x+(A-C) = x^2-x+2$$

Matching terms gives us $A+B=1$, $A-B+C=-1$, $A-C=2$, so $2A+C=0 \Rightarrow 2A=-C \Rightarrow A=2/3 \Rightarrow C=-4/3 \Rightarrow B=1/3$, then

$$I=\frac{2}{3}\int\frac{dx}{x-1}+\frac{1}{3}\int\frac{(x-4)dx}{x^2+x+1}=\frac{2}{3}\ln|x-1|+\frac{1}{3}\int\frac{(x+1/2)dx}{x^2+x+1}-\frac{1}{3}\int\frac{(9/2)dx}{x^2+x+1}.$$

But now we have to use u-sub on the second integral and use a clever algebraic trick on the third. For the first u-sub we have $u_1=x^2+x+1 \Rightarrow du_1=(2x+1)dx \Rightarrow (x+1/2)dx=du/2$, which gives us

$$I=\frac{2}{3}\ln|x-1|+\frac{1}{6}\int\frac{du_1}{u_1}-\frac{3}{2}\frac{dx}{(x+1/2)^2+3/4}.$$

Now we employ another u-sub, $u_2=x+1/2 \Rightarrow du_2=dx$,

$$I=\frac{2}{3}\ln|x-1|+\frac{1}{6}\ln|x^2+x+1|-\frac{3}{2}\frac{dx}{u_2^2+3/4}=\frac{2}{3}\ln|x-1|+\frac{1}{6}\ln|x^2+x+1|-2\frac{dx}{(2u_2/\sqrt{3})^2+1}$$

Then $v=2u_2/\sqrt{3} \Rightarrow dv=2du_2/\sqrt{3}$,

$$I=\frac{2}{3}\ln|x-1|+\frac{1}{6}\ln|x^2+x+1|-\sqrt{3}\frac{dx}{(v^2+1}=\frac{2}{3}\ln|x-1|+\frac{1}{6}\ln|x^2+x+1|-\sqrt{3}\tan^{-1}\left(\frac{2}{\sqrt{3}}(x+1/2)\right)+C.$$

Finally, we can have nondistinct quadratic factors

**Case 4**

Suppose $Q$ is a product of factors that include repeated quadratic terms. Then the repeated quadratic factors will be of the form

$$\frac{P(x)}{Q(x)}=\frac{P(x)}{(ax^2+bx+c)^r}=\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_rx+B_r}{(ax^2+bx+c)^r}. \tag{5}$$

An example of this is

Ex:

$$\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3}=\frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+x+1}+\frac{Ex+F}{x^2+1}+\frac{Gx+H}{(x^2+1)^2}+\frac{Ix+J}{(x^2+1)^3}.$$

Here is an example of Case 4

23) We break up the fraction as follows

$$\frac{Ay+B}{y^2+1}+\frac{Cy+D}{(y^2+1)^2} \Rightarrow (Ay+B)(y^2+1)+Cy+D = Ay^3+By^2+(A+C)y+(B+D)=y^2+2y+1.$$

Then we get $A=0$, $C=2$, $B=1$, and $D=0$

$$I=\int\frac{dy}{y^2+1}+\int\frac{2ydy}{(y^2+1)^2}=\tan^{-1}y+\int\frac{du}{u^2}=\tan^{-1}y-u^{-1}+C=\tan^{-1}y-\frac{1}{y^2+1}+C.$$

We can also have situations where we need to do long division – if the numerator is of a higher or equal order than the denominator.

33)

$$\frac{2x^3-2x^2+1}{x^2-x}=2x+\frac{1}{x^2-x}$$

Then our integral becomes

$$I=\int(2x+\frac{1}{x^2-x})dx=x^2+\int\frac{dx}{x-1}-\frac{dx}{x}=x^2+\ln|x-1|-\ln|x|+C.$$

We may also have to do u-sub before we can use partial fractions

39) Let $x = e^t \Rightarrow dx = e^t dt$, then

$$I = \int \frac{dx}{x^2+3x+2} = \int \frac{dx}{(x+2)(x+1)} = \int \frac{dx}{x+1} - \int \frac{dx}{x+2} = \ln|x+1| - \ln|x+2| + C = \ln(e^t+1) - \ln(e^t+2) + C.$$

Now lets look at some additional problems not in the book,

Ex: $I = \int \frac{x^3+x}{x-1} dx$,

**Solution:** By long division we get,

$$I = \int \left( x^2 + x + 2 + \frac{2}{x-1} \right) dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x + 2\ln|x-1| + C. \quad (6)$$

Ex: Convert $\frac{x^2+2x-1}{2x^3+3x^2-2x}$ into partial fractions.

**Solution:** First we factor out the denominator, then split the fractions

$$\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}.$$

Then

$$A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1) = (2A+B+2C)x^2 + (3A+2B-C)x - 2A = x^2 + 2x - 1$$

Matching coefficients gives us $A = 1/2$, then $B + 2C = 0 \Rightarrow B = -2C \Rightarrow C = -1/10 \Rightarrow B = 1/5$.

Ex: $I = \int \frac{x^4-2x^3+4x+1}{x^3-x^2-x+1} dx$.

**Solution:** We first use long division and then factor the denominator

$$\frac{x^4-2x^3+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1} = x+1+\frac{4x}{x^2(x-1)-(x-1)} = x+1+\frac{4x}{(x-1)^2(x+1)}.$$

Then we split up the fraction

$$\frac{4x}{x^3-x^2-x+1} = \frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

which gives us

$$A(x-1)(x+1) + B(x+1) + C(x-1)^2 = (A+C)x^2 + (B-2C)x + (-A+B+C) = 4x$$

Then we get $C = -A \Rightarrow B = -2C$, and $C = -1$, $A = 1$, $B = 2$. Our integral becomes

$$I = \int \left( x+1+\frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1} \right) dx = \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C$$

Ex: $I = \int \frac{2x^2-x+4}{x^3+x}$

**Solution:** Splitting the fractions gives us

$$\frac{2x^2-x+4}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \Rightarrow A(x^2+1) + Bx^2 + Cx = (A+B)x^2 + Cx + A = 2x^2 - x + 4$$

Then we get $A = 4$, $C = -1$ and $B = -2$. Our integral becomes

$$I = \int \left( \frac{1}{x} + \frac{x-1}{x^2+4} \right) dx = \ln|x| + \int \frac{xdx}{x^2+4} - \int \frac{dx}{x^2+4} = \ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}(x/2) + C.$$

Ex: $I = \int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}$

**Solution:** Splitting the fractions gives us

$$\frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} \Rightarrow A(x^2+1)^2 + (Bx+C)x(x^2+1) + Dx^2 + Ex$$

$$= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A = -x^3 + 2x^2 - x + 1.$$

Then we get $A = 1$, $B = -1$, $C = -1$, $D = 1$, $E = 0$, then our integral becomes

$$I = \int \left( \frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2} \right) dx = \ln|x| - \frac{1}{2}\ln|x^2+1| - \tan^{-1}x - \frac{1}{2(x^2+1)} + C.$$

Here are a couple of more problems that uses clever algebra

Ex: $I = \int \frac{4x^2-3x+2}{4x^2-4x+3}dx$

**Solution:** We first do long division

$$\frac{4x^2-3x+2}{4x^2-4x+3} = 1 + \frac{x-1}{4x^2-4x+3}$$

Notice that the denominator cannot be factored because its discriminant is $b^2 - 4ac = -32 < 0$. Recall that we can only factor quadratic polynomials whose discriminant is greater than or equal to zero. However we can complete the square on $4x^2 - 4x$,

$$I = x + \int \frac{x-1}{x^2-4x+3}dx = x + \int \frac{x-1}{(2x-1)^2+2}dx$$

We completed the square as follows

$$4x^2 - 4x = 4(x^2 - x) = 4\left(x^2 - x + \frac{1}{4}\right) - 1 = 4\left(x^2 - \frac{1}{2}\right)^2 - 1 = (2x^2 - 1) - 1.$$

Now we use u-sub where $u = 2x - 1 \Rightarrow du = 2dx$, then

$$I = x + \frac{1}{4}\int \frac{u-1}{u^2+2}dx = x + \frac{1}{4}\int \frac{udu}{u^2+2} - \frac{1}{4}\int \frac{du}{u^2+2} = x + \frac{1}{8}\ln|u^2+2| - \frac{1}{4\sqrt{2}}\tan^{-1}(u/\sqrt{2}) + C$$

$$= x + \frac{1}{8}\ln|(2x-1)^2+2| - \frac{1}{4\sqrt{2}}\tan^{-1}((2x-1)/\sqrt{2}) + C$$

Ex: $I = \int \frac{\sqrt{x+4}}{x}dx$.

**Solution:** We first use u-sub, $u^2 = x + 4 \Rightarrow 2udu = dx$,

$$I = 2\int \frac{u^2du}{u^2-4} = 2\int \left(1 + \frac{4}{u^2-4}\right)du = 2u + 2\int \frac{4}{(u-2)(u+2)}du = 2u + 2\int \frac{du}{u-2} + 2\int \frac{du}{u+2}$$

$$= 2u + 2\ln|u-2| - 2\ln|u+2| + C = 2\sqrt{x+4} + 2\ln|\sqrt{x+4}-2| - 2\ln|\sqrt{x+4}+2| + C$$

## 8.7 Numerical Integration

I will show three methods for solving integrals numerically. Note: midpoint rule isn't in the book, and therefore won't be on the exam.

Midpoint rule:

$$\int_a^b f(x)dx \approx \Delta x[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)]; x_i^* = \frac{1}{2}(x_i + x_{i+1}), \Delta x = \frac{b-a}{n} \quad (7)$$

Where $n$ is the number of intervals or equivalently the number of "steps".

$$\text{Error bound: } |E_M| \leq \frac{K(b-a)^3}{24n^2}; |f''(\xi)| \leq K, \xi \in [a,b]. \quad (8)$$

Where $|f''(\xi)|$ is just the maximum of the second derivative in $[a,b]$.

Ex: Consider the integral $I = \int_1^2 \frac{dx}{x}$. We note that the exact value of this integral is $I = \ln 2 \approx .693147$.

(a) Approximate the integral via Midpoint rule with $n = 5$ steps.

**Solution:** Here $a = 1$, $b = 2$, so $\Delta x = 1/5$. Also, clearly $x_i = a + i\Delta x$, so $x_0 = a = 1$, $x_1 = 1.2$, $x_2 = 1.4$, $x_3 = 1.6$, $x_4 = 1.8$, and $x_5 = b = 2$, so $x_1^* = 1.1$, $x_2^* = 1.3$, $x_3^* = 1.5$, $x_4^* = 1.7$, $x_5^* = 1.9$. Then plugging this into the formula gives,

$$I \approx \frac{1}{5}\left[\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right] \approx .691908$$

(b) Find the error bound for this approximation.
**Solution:** Notice that $b - a = 1$ and $n = 5$. Now we find $K$. To do this we take the second derivative $f''(x) = 2/x^3$. We notice that in $[1, 2]$ this is largest at $\xi = 1$, so $f''(\xi) = 2$. So, we choose $K = 2$. Plugging these into the formula gives,

$$|E_M| \leq \frac{K(b-a)^3}{24n^2} = \frac{2 \cdot 1}{24 \cdot 25} = \frac{1}{300}.$$

(c) Find the smallest $n$ that guarantees $|E_M| \leq 0.0001$.
**Solution:** This is a far more interesting problem. We start with the formula and put in the quantities we know,

$$\frac{K(b-a)^3}{24n^2} = \frac{1}{12n^2} \leq 0.0001 \Rightarrow n^2 \geq \frac{1}{0.0012} \Rightarrow n \geq \frac{1}{\sqrt{0.0012}} \approx 28.8$$

So, we need $n = 29$.

Trapezoid rule:

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]; \Delta x = \frac{b-a}{n}, x_i = a + i\Delta x. \quad (9)$$

Where $n$ is the number of intervals or equivalently the number of "steps".

$$\text{Error bound: } |E_T| \leq \frac{K(b-a)^3}{12n^2}; |f''(\xi)| \leq K, \xi \in [a, b]. \quad (10)$$

Where $|f''(\xi)|$ is just the maximum of the second derivative in $[a, b]$.

Ex: Take the same integral as in the Midpoint rule example, and answer the same exact questions.

(a) We already have the quantities we need from the Midpoint rule example, so we just plug those into the Trapezoid rule formula,

$$I \approx \frac{1}{10}\left[1 + 2\frac{1}{1.2} + 2\frac{1}{1.4} + 2\frac{1}{1.6} + 2\frac{1}{1.8} + \frac{1}{2}\right] \approx .695635$$

(b) For the error bound the difference between trapezoid rule and midpoint rule is a factor of 2, so plugging into the formula gives $|E_T| \leq 1/150$.

(c) We have the same quantities here, so we get $n > 40.8 \Rightarrow n = 41$.

Simpson's rule:

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}[f(x_0)+4f(x_1)+2f(x_2)+\cdots+4f(x_{n-1})+f(x_n)]; \Delta x = \frac{b-a}{n}, n \geq 4 \text{ and must be even.} \quad (11)$$

Where $n$ is the number of intervals or equivalently the number of "steps".

$$\text{Error bound: } |E_S| \leq \frac{K(b-a)^5}{180n^4}; |f^{(4)}(\xi)| \leq K, \xi \in [a, b]. \quad (12)$$

Where $|f^{(4)}(\xi)|$ is just the maximum of the fourth derivative in $[a, b]$.

Ex: Consider the same integral as the previous two examples.

(a) Approximate this integral with $n = 10$ steps.
**Solution:** Here $\Delta x = 1/10$, and $x_0 = 1$, $x_1 = 1.1$, $x_2 = 1.2$, $x_3 = 1.3$, $x_4 = 1.4$, $x_5 = 1.5$, $x_6 = 1.6$, $x_7 = 1.7$, $x_8 = 1.8$, $x_9 = 1.9$, and $x_{10} = 2$. Plugging these into the formula gives,

$$I \approx \frac{1}{30}\left[1 + 4\frac{1}{1.1} + 2\frac{1}{1.2} + 4\frac{1}{1.3} + 2\frac{1}{1.4} + 4\frac{1}{1.5} + 2\frac{1}{1.6} + 4\frac{1}{1.7} + 2\frac{1}{1.8} + 4\frac{1}{1.9} + \frac{1}{2}\right] \approx 0.693150$$

(b) Find the smallest $n$ that guarantees $|E_S| \le 0.0001$.
**Solution:** We have most of the quantities, so we must only look for $K$. Taking the fourth derivative gives $f^{(4)}(x) = 24/x^5$. We see that this is largest at $\xi = 1$ for our interval, so $f^{(4)}(\xi) = 24$, hence we choose $K = 24$. Plugging these into the formula gives

$$|E_S| \le \frac{24}{180n^4} \le 0.0001 \Rightarrow n^4 \ge \frac{24}{180(0.0001)} \Rightarrow n \ge \left(\frac{24}{180(0.0001)}\right)^{1/4} \approx 6.04.$$

So we have $n = 8$ because we need an even $n$.

## 8.8 Improper Integrals

Improper integrals are integrals that may blow up. This poses a question, what is infinity and how do we deal with it? Consider the following example

Ex: $\int_1^\infty dx/x^2$. We know how to integrate this for a finite interval, so why don't we do that and then take the limit to infinity.

$$\int_1^\infty \frac{dx}{x^2} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^2} = \lim_{t\to\infty} \left.\frac{-1}{x}\right|_1^t = \lim_{t\to\infty} 1 - \frac{1}{t} = 1.$$

We have two cases of improper integrals. One where the interval is infinite and another where the interval is finite but the integrand has a discontinuity.

**Case 1: Infinite Intervals**

a) If $\displaystyle\int_a^t f(x)dx$ exists for all $t \ge a$, then $\displaystyle\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$

b) If $\displaystyle\int_t^b f(x)dx$ exists for all $t \le b$, then $\displaystyle\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx.$

**Definition 1.** If $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are convergent if the limit exists, and divergent if the limit does not exist.

c) If $\displaystyle\int_a^\infty f(x)dx$ and $\displaystyle\int_{-\infty}^a f(x)dx$ are convergent, $\displaystyle\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx.$

Here are some examples we did in class

Ex:

$$\int_1^\infty \frac{dx}{x} = \lim_{t\to\infty} \int_1^t \frac{dx}{x} = \lim_{t\to\infty} \ln|x|\Big|_1^t = \lim_{t\to\infty} \ln|t| = \infty$$

So the integral diverges.

1)

$$I = \lim_{t\to\infty} \int_0^t \frac{dx}{1+x^2} = \lim_{t\to\infty} \tan^{-1} x\Big|_0^t = \frac{\pi}{2}$$

So the integral converges.

9)

$$I = \frac{1}{2}\lim_{t\to-\infty}\left[\int_t^{-2}\frac{dx}{x-1} - \int_t^{-2}\frac{dx}{x+1}\right] = \frac{1}{2}\lim_{t\to-\infty}\left[\ln|x-1| - \ln|x+1|\right]_t^{-2} = \frac{1}{2}\lim_{t\to-\infty}\left[\ln 3 - \ln|t-1| + \ln|t+1|\right]$$

$$= \frac{1}{2}\lim_{t\to\infty}\left[\ln 3 + \ln\left|\frac{t+1}{t-1}\right|\right] = \frac{1}{2}\lim_{t\to\infty}\left[\ln 3 + \ln\left|\frac{1+1/t}{1-1/t}\right|\right] = \frac{1}{2}\ln 3.$$

13)

$$I = \lim_{t\to\infty} \int_{-t}^{0} \frac{2x dx}{(x^2+1)^2} + \int_{0}^{t} \frac{2x dx}{(x^2+1)^2} = \lim_{t\to\infty} \left.\frac{-1}{x^2+1}\right|_{-t}^{0} + \left.\frac{-1}{x^2+1}\right|_{0}^{t} = \lim_{t\to\infty} -1 + \frac{1}{t^2+1} - \frac{1}{t^2+1} + 1 = 0.$$

Ex: For what values of $p$ is the integral $\int_1^\infty dx/x^p$ convergent?

**Solution:** Lets first look at $p \neq 1$. We can integrate and then deal with the two cases when we take the limit.

$$\int_1^\infty \frac{dx}{x^p} = \lim_{t\to\infty} \int_1^t x^{-p} dx = \lim_{t\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^t = \lim_{t\to\infty} \frac{1}{1-p}\left[\frac{1}{t^{p-1}} - 1\right].$$

If $p > 1$, $p-1 > 0$, then as $t \to \infty$, $t^{p-1} \to \infty$, so $1/t^{p-1} \to 0$, therefore $\int_1^\infty dx/x^p = 1/(p-1)$, and hence it converges. If $p < 1$, $p - 1 < 0$, so as $t \to \infty$, $1/t^{p-1} = t^{1-p} \to \infty$, therefore the integral diverges. And we know for $p = 1$, the integral diverges from a previous example.

We have just proved a theorem, which is stated below

**Theorem 1** (p-test)**.** *Consider $\int_1^\infty dx/x^p$. If $p > 1$, the integral converges to $1/(p-1)$, otherwise it diverges.*

Now, we move on to the second case, which is the case of finite intervals where the integrand has a discontinuity.

**Case 2: Integrands with Discontinuities**

a) If $f$ is continuous in $[a, b)$ and discontinuous at $x = b$, then $\displaystyle\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$.

b) If $f$ is continuous in $(a, b]$ and discontinous at $x = a$, then $\displaystyle\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$.

**Definition 2.** The integral $\int_a^b f(x)dx$ is said to be convergent if the limit exists, and divergent if the limit does not exist.

c) If $f$ has a discontinuity at $c \in [a, b]$ and $\displaystyle\int_a^c f(x)dx$ and $\displaystyle\int_c^b f(x)dx$ both converge, then $\displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

21)

$$\int_{-\infty}^{0} \theta e^\theta d\theta = \lim_{t\to-\infty} \int_t^0 \theta e^\theta d\theta = \left.\theta e^\theta - e^\theta + 1\right|_t^0 = \lim_{t\to-\infty} -\cancelto{0}{\theta e^\theta} + \cancelto{0}{e^\theta} - 1 = 1$$

So, the integral converges.

4)

$$I = \lim_{t\to 4} \int_0^t \frac{dx}{\sqrt{4-x}} = \lim_{t\to 4} \left.-\sqrt{4-x}\right|_0^t = \lim_{t\to 4} -\sqrt{4-t} + 2 = 2.$$

So the integral converges.

6) Here we have a discontinuity at $x = 0$, so we need to do two different integrals

$$\lim_{t\to 0} \int_t^1 x^{-1/3} dx = \lim_{t\to 0} \left.\frac{3}{2} x^{2/3}\right|_t^1 = \lim_{t\to 0} \frac{3}{2} - \frac{3}{2} t^{2/3} = \frac{3}{2}$$

$$\lim_{t\to 0} \int_{-8}^t x^{-1/3} dx = \lim_{t\to 0} \left.\frac{3}{2} x^{2/3}\right|_{-8}^t = \lim_{t\to 0} \frac{3}{2} t^{2/3} - 6 = -6$$

Since both the integrals converge, $I = -9/2$.

Ex: $\int_0^3 \frac{dx}{x-1}$.

**Solution:** Notice that there is a discontinuity at $x = 1$, so we need to split the integral,

$$\int_0^1 \frac{dx}{x-1} = \lim_{t\to 1} \int_0^t \frac{dx}{x-1} = \lim_{t\to 1} \ln|x-1|\Big|_0^t = \lim_{t\to 1}(\ln|t-1| - \ln|-1|) = -\infty$$

So this part of the integral diverges, which means the entire thing diverges. Now in class I showed why you can't just integrate it without taking care of the discontinuity. If you went ahead and integrated without splitting the integral you would get the wrong answer.

Sometime we can't integrate something, but we would still like to know the convergence. In these cases we employ comparison tests.

**Direct Comparison Test(DCT):** If $f, g$ are continuous with $f \geq g \geq 0$ for $x \geq a$,

$$\text{a)} \int_a^\infty f(x)dx \text{ converges } \Rightarrow \int_a^\infty g(x)dx \text{ converges.}$$

$$\text{b)} \int_a^\infty g(x)dx \text{ diverges } \Rightarrow \int_a^\infty f(x)dx \text{ diverges.}$$

- If the bigger one converges, so does the smaller one.
- If the smaller one diverges, so does the bigger one.

Now lets look at an integral that is impossible to solve with the techniques we know currently

Ex: $\int_0^\infty e^{-x^2}dx$

**Solution:** Notice $e^{-x} \geq e^{-x^2}$ for $x \geq 1$, so we can integrate this

$$\int_1^\infty e^{-x}dx = \lim_{t\to\infty} \int_1^t e^{-x}dx = \lim_{t\to\infty}(e^{-1} - e^{-t}) = e^{-t}.$$

Since the bigger one converges, $\int_0^\infty e^{-x^2}dx$ converges by DCT.

Next we look at a few book problems,

47)

$$\frac{1}{x^3} \geq \frac{1}{x^3+1} \text{ and } \int_1^\infty \frac{dx}{x^3} \text{ converges since } p > 1, \text{ so } \int_1^\infty \frac{dx}{x^3+1} \text{ converges as well.}$$

35) Here we use u-sub with $u = \cos\theta \Rightarrow du = \sin\theta d\theta$

$$\int_0^{\pi/2} \tan\theta d\theta = \int_1^\infty \frac{du}{u} \text{ diverges since } p = 1.$$

39) We again employ u-sub with $u = 1/x \Rightarrow du = -1/x^2$

$$\int_0^{\ln 2} x^{-2}e^{-1/x}dx = \int_{1/\ln 2}^\infty e^{-u}du = -e^{-u}\Big|_{1/\ln 2}^\infty = e^{-1/\ln 2}$$

So the integral converges.

**Limit Comparison Test(LCT):** If $f, g$ are continuous and $\lim_{x\to\infty} f(x)/g(x) = L$, then either $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ both converge or both diverge.

We only want to use this test if it is absolutely necessary.

41) We compare to $1/\sqrt{t}$ because $\sin t \sim t$ as $t \to \infty$. Taking the limit gives us

$$\lim_{t\to 0} \frac{f(x)}{g(x)} = \lim_{t\to 0} \frac{\sqrt{t}}{\sqrt{t}+\sin t} = \lim_{t\to 0} \frac{1}{1+t/\sqrt{t}} = 1. \qquad \checkmark$$

This shows that our guess was correct. Now,

$$\int_0^\pi \frac{dt}{\sqrt{t}} = 2\sqrt{t}\Big|_0^\pi = 2\pi$$

So the compared integral converges, then by LCT the original integral also converges.

59) Here we will notice that $e^x/x \geq 1/x$, so we can simply do a direct comparison and $\int_1^\infty dx/x$ diverges since $p = 1$, so the original integral also diverges.

55) Here we have a cosine in the numerator, so we can bound it from both sides

$$\frac{1}{x} \leq \frac{2+\cos x}{x} \leq \frac{3}{x}$$

And we know that $\int_1^\infty dx/x$ diverges, so lets us the lower bound to compare. Since $\int_\pi^\infty dx/x$ diverges because $p = 1$, the original integral also diverges.

Now lets look at some addition examples that aren't in the book,

Ex: $\int_2^5 dx/\sqrt{x-2}$.

**Solution:** Our discontinuity is at $x = 2$, so

$$\int_2^5 \frac{dx}{\sqrt{x-2}} = \lim_{t\to 2} \int_t^5 \frac{dx}{\sqrt{x-2}} = \lim_{t\to 2} 2\sqrt{x-2}\Big|_t^5 = \lim_{t\to 2} 2(\sqrt{3}-\sqrt{t-2}) = 2\sqrt{3}.$$

So the integral converges.

Ex: $\int_0^{\pi/2} \sec x dx$.

**Solution:** The discontinuity is at $x = \pi/2$, so

$$\int_0^{\pi/2} \sec x dx = \lim_{t\to\pi/2} \int_0^t \sec x dx = \lim_{t\to\pi/2} \ln|\sec x + \tan x|\Big|_0^t = \lim_{t\to\pi/2} \ln|\sec t + \tan t| = \infty$$

So the integral diverges.

Ex: $\int_0^1 \ln x dx$.

**Solution:** Here the discontinuity is at $x = 1$, and we also must integrate by parts with $u = \ln x \Rightarrow du = 1/x$ and $dv = dx \Rightarrow v = x$,

$$\int_0^1 \ln x dx = \lim_{t\to 0} \int_t^1 \ln x dx = \lim_{t\to 0} x\ln x\Big|_t^1 - \int_t^1 dx = \lim_{t\to 0} -t\ln t - 1 + t.$$

We deal with the first limit using L'Hôpital's rule

$$\lim_{t\to 0} t\ln t = \lim_{t\to 0} \frac{\ln t}{1/t} = \lim_{t\to 0} \frac{1/t}{-1/t^2} = 0.$$

Therefore the integral converges to $\int_0^1 \ln x dx = -1$.

Ex: $\int_1^\infty [(1+e^{-x})/x] dx$.

**Solution:** Notice this integral would be a pain to evaluate, but we get a feeling that it diverges. Now, $\frac{1+e^{-x}}{x} \geq \frac{1}{x}$ because the exponential function is always positive. Now we know that $\int_1^\infty dx/x$ diverges because $p = 1$. Therefore, by DCT the original integral diverges.

Ex: $\int_1^\infty \frac{\sqrt{x^2+1}}{x^3}dx$.

**Solution:** Here the root gives us some difficulty in finding a direct comparison, but we can find something that would work for limit comparison. Since the problem is in the numerator, lets divide through by the highest power in the numerator

$$\frac{\sqrt{x^2+1}}{x^3} = \frac{\sqrt{1+1/x^2}}{x^2} \sim \frac{1}{x^2}.$$

Now we must take the limit of the ratios to prove that this is a valid comparison

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}/x^3}{1/x^2} = \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} = \lim_{x\to\infty} \sqrt{1+\frac{1}{x^2}} = 1.$$

Since this is a valid comparison, and we know $\int_1^\infty dx/x^2$ converges because $p > 1$, by LCT the original integral also converges.

Ex: $\int_1^\infty \frac{x}{\sqrt{x^3+2}}dx$.

**Solution:** Here the problem is with the denominator, so lets divide through by the highest power of the denominator

$$\frac{x}{\sqrt{x^3+2}} = \frac{x/\sqrt{x^3}}{\sqrt{x^3+2}/\sqrt{x^3}} = \frac{1/\sqrt{x}}{1+2/x^3} \sim \frac{1}{\sqrt{x}}.$$

Now we take the limit of the ratios

$$\lim_{x\to\infty} \frac{x/\sqrt{x^3+2}}{1/\sqrt{x}} = \lim_{x\to\infty} \frac{\sqrt{x^3}}{\sqrt{x^3+2}} = \lim_{t\to\infty} \frac{1}{\sqrt{1+2/x^3}} = 1.$$

Now, $\int_1^\infty dx/\sqrt{x}$ diverges because $p < 1$, so by LCT the original integral also diverges.

## 10.1 Sequences

Sequences are just functions, except as opposed to standard functions whose domains are the real numbers, the domain for sequences are the integers. So, we can think of them as regular functions, but we must be careful in certain instances.

Lets quickly go thorugh a few different ways of representing a sequence,

a) $\{\frac{n}{n+1}\}_{n=1}^\infty$; $a_n = \frac{n}{n+1}$; $\{\frac{1}{2}, \frac{2}{3}, \cdots, \frac{n}{n+1}, \cdots\}$

b) $\{(-1)^n \frac{n+1}{3^n}\}_{n=1}^\infty$; $a_n = (-1)^n \frac{n+1}{3^n}$; $\{-\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \cdots, (-1)^n \frac{n+1}{3^n}, \cdots\}$

c) $\{\sqrt{n-3}\}_{n=3}^\infty$; $a_n = \sqrt{n-3}, n \geq 3$; $0, 1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n-3}, \ldots$

d) $\{\cos\frac{n\pi}{6}\}_{n=0}^\infty$; $a_n = \cos\frac{n\pi}{6}, n \geq 0$; $1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \ldots, \cos\frac{n\pi}{6}, \cdots$

An important skill to have is deriving a general formula for a sequence from looking at a few terms of the sequence,

Ex: Find the formula for $\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \cdots\}$.

**Solution:** The first thing we notice is that there is an alternating sign, and since the first element ($n = 1$) is positive, we need to start off with an even power of $-1$, so $(-1)^{n-1}$ works. Notice we could have also used $(-1)^{n+1}$. We also notice that the denominators are respective powers of 5, so the denominator must be $5^n$. Now, we notice that the numerator starts with 3 and goes up by one every time, so the numerator in $n+2$, then $a_n = (-1)^{n-1}\frac{n+2}{5^n}$. Notice that we did this differently than in class. This was just to show another way to do the problem, but I actually prefer the way we did it in class.

We can also take limits of sequences, which is what we are most interested in for this class Lets take limits of the sequences from our first example.

$$\text{a) } \lim_{n\to\infty} \frac{n}{n+1} = 1 \qquad \text{b) } \lim_{n\to\infty} (-1)^n \frac{n+1}{3^n} = 0 \qquad \text{c) } \lim_{n\to\infty} \sqrt{n-3} = \infty$$

$$\text{d) } \lim_{n\to\infty} \cos\frac{n\pi}{6} \text{DNE} \qquad \text{e) } \lim_{n\to\infty} (-1)^{n-1}\frac{n+2}{5^n} = 0.$$

Just as with real valued functions we can define convergence and divergence,

**Definition 3.** If $\lim_{n\to\infty} a_n = L$ we say it is convergent, otherwise it is divergent.

Lets remind ourselves of the standard limit laws,

**Theorem 2.** *If $\{a_n\}$ and $\{b_n\}$ are convergent sequences, then*

$$\begin{aligned}
&a)\ \lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n \\
&b)\ \lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n \\
&c)\ \lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) \\
&d)\ \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \text{ if } \lim_{n\to\infty} b_n \neq 0 \\
&e)\ \lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty a_n}\right]^p \text{ if } p > 0 and a_n \geq 0.
\end{aligned}$$

We also have the squeeze theorem and a very important consequence of the squeeze theorem,

**Theorem 3.** *If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.*

**Theorem 4.** *If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.*

Lets do a few easy examples before getting to the more difficult ones

$$35)\ \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0 \qquad \text{Ex)}\ \lim_{n\to\infty} (-1)^n \text{ DNE} \qquad \text{Ex)}\ \lim_{n\to\infty} \frac{(-1)^n}{n} = 0.$$

Here is a problem that is quite a bit tougher and requires some clever algebra,

63) Because this is tricky, lets write down the first few terms of the $n^{\text{th}}$ element of the sequence, $a_n = \frac{1\cdot 2\cdot 3\cdots n}{n\cdot n\cdot n\cdots n}$. Now, we get a better idea of what's going on. Lets factor out $1/n$, since we know what happens to that sequence. If we do this, we notice $\frac{2\cdot 3\cdots n}{n\cdot n\cdots n} \leq 1$ for $n \geq 1$. Then our sequence is always positive, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1\cdot 2\cdots n}{n\cdot n\cdots n} = \lim_{n\to\infty} \frac{1}{n}\left(\frac{2\cdot 3\cdots n}{n\cdot n\cdots n}\right) \leq \lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefor our sequence too converges to 0.

Now lets think of what happens to the sequence $\{r^n\}$. If $|r| < 0$, $\lim_{n\to\infty} r^n = 0$. If $|r| > 1$, $\lim_{n\to\infty} r^n = \infty$. If $r = 1$, $\lim_{n\to\infty} r^n = 1$, and if $r = -1$ we know it doesn't exist by a previous example. So, the series converges for $|r| < 1$ and $r = 1$.

**Monotonic Sequences.** What if we couldn't take the limit of a sequence, but we knew some properties of the function and wanted to analyze the behavior. The following definitions and theorem will help us deal with this.

**Definition 4.** A sequence $a_n$ is called increasing (also called nondecreasing) if $a_n \leq a_{n+1}$ for all $n \geq 1$, i.e. $a_1 \leq a_2 \leq a_3 \leq \cdots$. It is called decreasing (also called nonincreasing) if $a_n \geq a_{n+1}$ for all $n \geq 1$, i.e. $a_1 \geq a_2 \geq a_3 \geq \cdots$. These types of sequences are collectively called monotonic sequences.

Lets look at a few examples of monotonic sequences,

$$\text{Ex: } \left\{\frac{3}{n+5}\right\} \text{ is decreasing because } \frac{3}{n+5} > \frac{3}{(n+1)+5} = \frac{3}{n+6}.$$

$$\text{Ex: } \left\{\frac{n}{n^2+1}\right\}_{n=1}^{\infty} \text{ is decreasing because } \left(\frac{n}{n^2+1}\right)' = \frac{1-n^2}{(n^2+1)^2} < 0 \text{ for } n > 1$$

Now if the sequence is bounded we can say something about its convergence.

**Definition 5.** A sequence $a_n$ is said to be bounded above if there is an $M$ such that $a_n \leq M$ for all $n \geq 1$, and bounded below if there is an $m$ such that $a_n \geq m$ for all $n \geq 1$.

**Theorem 5.** *Every (respectively) bounded monotonic sequence is convergent.*

We can also have sequences where the next term depends directly on the previous term(s). These are called recurrence relations and are of the form $a_{n+1} = f(a_n, a_{n-1}, \ldots, a_1)$. Consider the simple recurrence relation $a_{n+1} = (a_n + 6)/2$. Notice that if the limit exists, $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = a_*$. Here, $a_*$ is called a fixed point. Now we can plug this in and find the value for it, $a_* = \frac{1}{2}(a_* + 6) \Rightarrow a_* = 6$.

Now lets do some problems from the book

46) Notice $\lim_{n\to\infty} \left|\frac{\sin^2 n}{2^n}\right| \leq \lim_{n\to\infty} \frac{1}{2^n} = 0$.

47) $\lim_{n\to\infty} \frac{n}{2^n} = \lim_{n\to\infty} \frac{1}{(\ln 2)e^{n\ln 2}} = 0$.

68)

$$\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\ln(1+1/n)}{1/n} = \lim_{n\to\infty} \frac{-1/(n^2+n)}{-1/n^2} = \lim_{n\to\infty} \frac{n^2}{n^2+n} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1.$$

54)

$$\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n = \lim_{n\to\infty} e^{\ln(1-1/n)^n} = \exp\left[\lim_{n\to\infty} n\ln\left(1-\frac{1}{n}\right)\right]$$

and

$$\lim_{n\to\infty} \frac{\ln(1-1/n)}{1/n} = \lim_{n\to\infty} \frac{(1/n^2)/(1-1/n)}{-1/n^2} = \lim_{n\to\infty} \frac{-1}{1-1/n} = -1 \Rightarrow \lim_{n\to\infty} a_n = e^{-1}$$

71) Again we use the $e^{\ln}$ trick,

$$\lim_{n\to\infty} \left(\frac{x^n}{2n+1}\right)^{1/n} = \lim_{n\to\infty} e^{\ln(x^n/(2n+1))^(1/n)} = \exp\left[\lim_{n\to\infty} \frac{1}{n}\ln\left(\frac{x^n}{2n+1}\right)\right].$$

Then we take the limit

$$\lim_{n\to\infty} \frac{\ln(x^n/(2n+1))}{n} = \lim_{n\to\infty} \frac{\ln(x^n)-\ln(2n+1)}{n} = \lim_{n\to\infty} \frac{n\ln x - \ln(2n+1)}{n}$$
$$= \lim_{n\to\infty} \frac{\ln x - 2/(2n+1)}{1} = \lim_{n\to\infty} \ln x - \frac{2}{2n+1} = \ln x.$$

Plugging this back in gives $e^{\ln x} = x$.

72) Again,

$$\lim_{n\to\infty} \left(1-\frac{1}{n^2}\right)^n = \lim_{n\to\infty} e^{\ln(1-1/n^2)^n} = \exp\left[\lim_{n\to\infty} n\ln\left(1-\frac{1}{n^2}\right)\right].$$

Taking the limit gives us

$$\lim_{n\to\infty} \frac{\ln(1-1/n^2)}{1/n} = \lim_{n\to\infty} \frac{-2/(n-n^3)}{-1/n^2} = \lim_{n\to\infty} \frac{2n^2}{n-n^3} = \lim_{n\to\infty} \frac{2}{1/n-n} = 0.$$

Plugging this back in gives us $e^0 = 1$.

84) And once again,

$$\lim_{n\to\infty} e^{\ln(n^2+n)^{1/n}} = \exp\left[\lim_{n\to\infty} \frac{1}{n}\ln(n^2+n)\right]$$

The limit gives us

$$\lim_{n\to\infty} \frac{\ln(n^2+n)}{n} = \lim_{n\to\infty} \frac{(2n+1)/(n^2+n)}{1} = \lim_{n\to\infty} \frac{2n+1}{n^2+n} = \lim_{n\to\infty} \frac{2+1/n}{n+1} = 0$$

Plugging this back in gives us $e^0 = 1$.

90) We have done a problem like this in the improper integral section. This reiterates the intimate relationship between integrals and sequences. $\lim_{n\to\infty} \int_1^n dx/x^p$, for $p > 1$ converges to $1/(p-1)$. We can see this by integrating it.