10.2 Infinite Series

A series is a sum of sequential terms. An infinite series can be represented as such: $\sum_{n=1}^{\infty} a_n$. We also think of series as a sequence of partial sums, where each partial sum is $s_N = \sum_{n=1}^{N} a_n$. We have to make sure we don't confuse these very different sequences. One is a sequence that is being summed, and the other is a sequence of sums.

Definition 1. Given $\sum_{n=1}^{\infty} a_n$, let $s_n = \sum_{i=1}^{n} a_i$ be the partial sums. If s_n converges and $\lim_{n\to\infty} s_n = s$ exists, then we say $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} a_n = s$. Otherwise, we say it diverges.

Ex: Consider the series $\sum_{n=1}^{\infty} a \cdot r^{n-1}$. This is a very important series called the geometric series. What does this converge to?

Notice if $r = 1$, $s_n = a + a + \cdots + a = na \to \pm\infty$, so it diverges. Now, if $r = -1$, the partial sum will jump between zero and one, so it also diverges.

If $|r| \neq 1$, $s_n = a + ar + ar^2 + \cdots + ar^{n-1}$, and $rs_n = ar + ar^2 + \cdots + ar^n$, then $s_n - rs_n = a - ar^n \Rightarrow s_n = \frac{a(1-r^n)}{1-r}$. Now, for $-1 < r < 1$, $r^n \to 0$ as $n \to \infty$, hence $\lim_{n\to\infty} s_n = a/(1-r)$. For $|r| > 1$, $r^n \to \infty$, so s_n clearly diverges.

Theorem 1. *The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges for $|r| < 1$ to*

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \tag{1}$$

and diverges otherwise.

Ex: Find the sum of $S = 5 - 10/3 + 20/9 - 40/27 + \cdots$.

Solution: Notice that we can immediately factor out a 5, $S = 5[1 - 2/3 + 4/9 - 8/27 + \cdots]$. Now we notice that we have alternating signs, so we must have a $(-1)^{n-1}$ because the first term is positive (if the first term was negative it would be $(-1)^n$). Next, we notice that all the terms are powers of $2/3$, via the geometric series theorem our sum is

$$\sum_{n=1}^{\infty} 5\left(-\frac{2}{3}\right)^{n-1} = \frac{5}{1+2/3} = \frac{5}{5/3} = 3.$$

Ex: Is $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent or divergent?

Solution: This series isn't in the form of the geometric series, so we must convert it into that form,

$$\sum_{n=1}^{\infty} 2^{2n}3^{1-n} = \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}.$$

This does not converge because $|r| = 4/3 > 1$, so it violates the hypothesis of the geometric series theorem.

Ex: Write $2.3\overline{17}$ as a geometric series.

Solution: We must this as a constant plus a fraction,

$$2.3\overline{17} = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \cdots = 2.3 + 17\left[\frac{1}{10^3} + \frac{1}{10^5} + \cdots\right] = 2.3 + 17\sum_{n=1}^{\infty}\left(\frac{1}{10}\right)^{2n+1}.$$

Ex: For what values of x does $\sum_{n=0}^{\infty} x^n$ (this is called a power series) converge?

Solution: This is exactly a geometric series if x were fixed. Now we may not be able to see this right away, but if we play around with the index we get

$$\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}; |x| < 1. \tag{2}$$

Ex: Telescoping series: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Notice this looks a lot like a partial fraction, so $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. So we get,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1} \to 1 \text{ as } n \to \infty. \tag{3}$$

Ex: Harmonic series: $\sum_{n=1}^{\infty} 1/n$.

Notice that the sequence $1/n$ converges to 0 as $n \to \infty$, however we will show that the series diverges. In order to do this we calculate the partial sums and put estimates on them, $s_1 = 1$, $s_2 = 1 + 1/2$, $s_4 = 1 + 1/2 + (1/3 + 1/4) > 1 + 1/2 + (1/4 + 1/4) = 2$, $s_8 > 1 + 3/2$, $s_{16} > 1 + 4/2$, $s_{32} > 1 + 5/2$, $s_{64} > 1 + 6/2$, so $s_{2^n} > 1 + n/2 \to \infty$ as $n \to \infty$. So, by definition, the series diverges.

The following two theorems give us a framework to prove divergence, but NOT convergence.

Theorem 2. *If the series* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

Proof. We can calculate the partial sums,

$$s_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$$
$$s_{n-1} = a_1 + a_2 + \cdots + a_{n-2} + a_{n-1}$$

Now, if we subtract the two, we get $s_n - s_{n-1} = a_n$, so we have a representation of a_n from the partial fractions. Now, since the series converges, the partial sums converge to exactly that sum, so $\lim_{n\to\infty} s_n = s$ and $\lim_{\to\infty} s_{n-1} = s$. Therefore, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - s_{n-1} = s - s = 0$. □

Corollary 1. *If* $\lim_{n\to\infty} a_n \neq 0$ *or does not exist, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Ex: Show $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges

Solution: We can just show that the sequence a_n does not go to zero.

$$\lim_{n\to\infty} \frac{n^2}{5n^2+4} = \lim_{n\to\infty} \frac{1}{5 + 4/n^2} = \frac{1}{5} \neq 0.$$

We also have some of the usual arithmetic properties for sums.

Theorem 3. *If* $\sum a_n$ *and* $\sum b_n$ *converge,* $\sum ca_n$ *and* $\sum a_n \pm b_n$ *converge, and*

$$a) \sum ca_n = c \sum a_n \text{ and } b) \sum (a_n \pm b_n) = \sum a_n \pm \sum b_n. \tag{4}$$

Ex: First we find the two sums individually,

$$\sum_{n=1}^{\infty} \frac{3}{n(n+1)} = 3 \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = 3.$$

and

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} = \frac{1/2}{1 - 1/2} = 1.$$

So, the series converges to

$$\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right) = 4.$$

10.3 Integral Test

I provided the motivation for integral test in class, so here I'll just go over the test itself.

Ex: Lets look at $\sum_{n=1}^{\infty} 1/n^2$. If we look at the partial sum we have $\lim_{n\to\infty} s_n < 1 + \int_1^\infty dx/x^2$ because the partial sums are right hand Reimann sums. So, if the integral converges the series will also converge. But we already know the integral converges since $p > 1$. So, the series too converges.

We have a similar result for series that diverge, but let's not go over that and get straight to the test.

Theorem 4 (Integral test). *Suppose f is continuous, positive, and decreasing on $[1, \infty)$ and let $a_n = f(n)$, then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the integral $\int_1^\infty f(x)dx$ also converges, i.e.*

$$\int_1^\infty f(x)dx \text{ converges } \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges. } \int_1^\infty f(x)dx \text{ diverges } \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges.} \tag{5}$$

Ex: Does the series $\sum_{n=1}^{\infty} 1/(n^2+1)$ converge?

Solution: We integrate

$$\int_1^\infty \frac{dx}{x^2+1} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^2+1} - \lim_{t\to\infty} \tan^{-1} x\Big|_1^t = \lim_{t\to\infty} \left(\tan^{-1} t - \frac{\pi}{4}\right) = \frac{\pi}{4}.$$

Since the integral converges, so does the series by integral test.

Ex: For what values of p does $\sum_{n=1}^{\infty} 1/n^p$ converge?

Solution: We have to integrate $\int_1^\infty dx/x^p$, but we already know this converges for $p > 1$, and diverges otherwise. Therefore, by integral test, the series too converges for $p > 1$, and diverges otherwise. This is called a *p-series.*

Theorem 5 (p-series). *The series $\sum_{n=1}^{\infty} 1/n^p$ converges for $p > 1$, and diverges otherwise.*

Ex: $\sum_{n=1}^{\infty} 1/n^3$ converges by p-test since $p = 3 > 1$.

Ex: $\sum_{n=1}^{\infty} 1/n^{1/3}$ diverges by p-test since $p = 1/3 < 1$.

Ex: Does $\sum_{n=1}^{\infty} (\ln n)/n$ converge?

Solution: We have to integrate this,

$$\int_1^\infty \frac{\ln x}{x} dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x} dx = \lim_{t\to\infty} \frac{1}{2} (\ln x)^2 \Big|_1^t = \lim_{t\to\infty} \frac{1}{2} (\ln t)^2 = \infty$$

Since the integral diverges, the series will also diverge by integral test.

In real life there may be times when we won't be able to find the sum of certain convergent series. In these cases it is beneficial to estimate the sum. Notice the bigger partial sum we take, the better the estimate, but how can we tell how good it is? Since it converges, we can use two integrals to do this. Notice that $s \le s_n + \int_n^\infty f(x)dx$ and $s \ge s_n + \int_{n+1}^\infty f(x)dx$ because these are left and right hand Riemann estimates for integrals of monotonic functions.

Definition 2. Suppose $\sum_{n=1}^{\infty} a_n = s$, and s_n are it's partial sums. Then the <u>remainder</u> of the n^{th} partial sum is $R_n = s - s_n$.

Theorem 6 (Remainder). *Consider $\sum_{n=1}^{\infty} a_n = s$. Suppose $f(x) = a_k$, where f is continuous, positive, and decreasing for $x \ge n$, then*

$$\int_{n+1}^\infty f(x)dx \le R_n \le \int_n^\infty f(x)dx. \tag{6}$$

Ex: Consider $\sum_{n=1}^{\infty} 1/n^3$.

(a) Find the maximum error for s_{10}.

Solution: We just plug this into the formula,

$$R_{10} \le \int_{10}^{\infty} \frac{dx}{x^3} = \lim_{t\to\infty} \int_{10}^{t} \frac{dx}{x^3} = \lim_{t\to\infty} \frac{-1}{2x^2}\bigg|_{10}^{t} = \lim_{t\to\infty} \frac{-1}{2t^2} - \frac{-1}{2\cdot 10^2} = \frac{1}{200} = 0.005$$

(b) How many terms must we take for $R_n \le 0.0005$?

Solution: Here we bound our formula and see what n has to be,

$$R_n \le \int_{n}^{\infty} \frac{dx}{x^3} = \frac{1}{2n^3} < 0.0005 \Rightarrow n^2 > \frac{1}{0.001} = 1000 \Rightarrow n > \sqrt{1000} \approx 31.6$$

So, we must take $n = 32$ terms.

(c) Now, notice if we add s_n to both sides of the inequality we get bounds on the exact solution, i.e. $s_{10} \approx 1.1975$, so for $n = 10$,

$$s_{10} + \int_{11}^{\infty} f(x)dx \le R_{10} + s_{10} \le s_{10} + \int_{10\infty} f(x)dx$$

$$\Rightarrow 1.1975 + \frac{1}{242} \le s \le 1.1975 + \frac{1}{200} \Rightarrow 1.2016 \le \sum_{n=1}^{\infty} \frac{1}{n^3} \le 1.2025$$

10.4 Comparison Tests

This is very similar to integral comparison tests.

Theorem 7 (Direct Comparison)**.** *Suppose $\sum a_n$ and $\sum b_n$ have positive terms, then*

(i) *If $\sum b_n$ converges and $a_n \le b_n$ for all $n > N$ for some N, then $\sum a_n$ also converges.*
(ii) *If $\sum b_n$ diverges and $a_n \ge b_n$ for all $n > N$ for some N, then $\sum a_n$ also diverges.*

State whether or not the following converge/diverge, and state why.

Ex: $\sum_{n=1}^{\infty} 1/(2^n + 1)$.

Solution: We know $\frac{1}{2^n+1} < \frac{1}{2^n}$. Therefore, since $\sum_{n=1}^{\infty} 1/2^n$ converges by p-test because $p > 1$, $\sum_{n=1\infty} 1/(2^n + 1)$ also converges by direct comparison.

Ex: $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$.

Solution: Here as usual we take the highest power of the top and bottom. This will give us $5/2n^2$. We know the sum of this converges, so for direct comparison we would attempt to show that this is greater than our original sequence. This is easy to show sine all the terms in the denominator are additive, so $\frac{5}{2n^2+4n+3} \le \frac{5}{2n^2}$. Since $\frac{5}{2}\sum_{n=1}^{\infty} 1/n^2$ converges by p-test because $p > 1$, the original series also converges by direct comparison.

Ex: $\sum_{n=1}^{\infty} (\ln n)/n$.

Solution: Since $\ln n > 1$ for $n \ge 3$, $\frac{\ln n}{n} \ge \frac{1}{n}$ for $n \ge 3$. Further, since $\sum_{n=1}^{\infty} 1/n$ diverges by p-test because $p = 1$, by direct comparison, the original series converges as well. Notice that we only care about the tail end.

Notice that we can't use this test on something like $\sum_{n=1}^{i} nfty1/(2^n - 1)$, so we need the limit comparison test.

Theorem 8 (Limit Comparison)**.** *Suppose $\sum a_n$ and $\sum b_n$ have positive terms, and $\lim_{n\to\infty} a_n/b_n = c > 0$, where c is a finite number. Then, either both $\sum a_n$ and $\sum b_n$ converges or both diverge. Further, if $c = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges, and if $c = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.*

State whether the following converge or diverge, and state the reasoning

Ex: $\sum_{n=1}^{\infty} 1/(2^n - 1)$.

Solution: Again, we take the highest power of both the top and bottom; i.e. $1/2^n$. Taking the limit gives,

$$\lim_{n\to\infty} \frac{1/(2^n-1)}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{1}{1-1/2^n} = 1\checkmark$$

Since $\sum_{n=1}^{\infty} 1/2^n$ converges by geometric series because $|r| = 1/2 < 1$, by the limit comparison test, $\sum_{n=1}^{\infty} 1/(2^n - 1)$ also converges.

Ex: $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$.

Solution: In class I showed the proper way to do this, but you can also think about it as taking the largest power of the numerator and denominator; i.e. $2n^2/n^{5/2} = 2/\sqrt{n}$. Taking the limit gives,

$$\lim_{n\to\infty} \frac{(2n^2+3n)/\sqrt{5+n^5}}{2/\sqrt{n}} = \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{2\sqrt{5+n^5}} = \lim_{n\to\infty} \frac{2+3/n}{2\sqrt{5/n^5+1}} = 1\checkmark$$

Since $\sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges by p-test because $p < 1$, by the limit comparison test, $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ also diverges.

Here are some book problems we did in class:

19) Notice,

$$0 \le \frac{\sin^2 n}{2^n} \le \frac{1}{2^n}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges by geometric series test because $|r| = 1/2 < 1$, the original series also converges by DCT.

21) Lets take the limit,

$$\lim_{n\to\infty} \frac{2n}{3n-1} = \lim_{n\to\infty} \frac{2}{3-1/n} = \frac{2}{3} \neq 0$$

So, the series diverges by the n^{th} term test.

25) Notice

$$\left(\frac{n}{3n+1}\right)^n = \left(\frac{1}{3+1/n}\right)^n < \left(\frac{1}{3}\right)^n.$$

Since $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ converges by geometric series test because $|r| = 1/3 < 1$, the original series converges by DCT.

31) Notice

$$\frac{1}{1+\ln n} \ge \frac{1}{n}.$$

Since $\sum_{n=1}^{\infty} 1/n$ diverges by p-test because $p = 1$, the original series diverges by DCT.

40) Here we can just take the highest term in the numerator and denominator,

$$\frac{2^n+3^n}{3^n+4^n} \sim \frac{3^n}{4^n} = \left(\frac{3}{4}\right)^n.$$

Now, lets take the limit of the two ratio of the two functions,

$$\lim_{n\to\infty} \frac{(2^n+3^n)/(3^n+4^n)}{3^n/4^n} = \lim_{n\to\infty} \frac{8^n+12^n}{9^n+12^n} = \lim_{n\to\infty} \frac{(8/12)^n+1}{(9/12)^n+1} = 1\checkmark$$

Since $\sum_{n=1}^{\infty} (3/4)^n$ converges by geometric series test because $|r| = 3/4 < 1$, the original series converges by LCT.

10.5 Ratio and Root Tests

Sometimes we need to bring out the big guns.

Theorem 9 (Ratio Test). *Consider* $\sum a_n$*, and suppose* $\lim_{n\to\infty} |a_{n+1}/a_n| = L$*, then*

a) *If* $L < 1$*, then* $\sum a_n$ *converges absolutely,*
b) *If* $L > 1$*, then* $\sum a_n$ *diverges,*
c) *and if* $L = 1$*, the test is inconclusive.*

State whether the following converge/diverge, and state why.

Ex: $\sum_{n=1}^{\infty} n^3/3^n$.

Solution: We apply the ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}\right| = \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1+\frac{1}{n}\right)^3.$$

Taking the limit gives,

$$\lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^3 = \frac{1}{3} < 1.$$

Therefore, by the ratio test, the series converges absolutely.

Ex: $\sum_{n=1}^{\infty} n^n/n!$.

Solution: We apply the ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}\right| = \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n.$$

Taking the limit gives

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \exp\left(\lim_{n\to\infty} n \ln\left(1+\frac{1}{n}\right)\right),$$

Now, we just look at the inside

$$\lim_{n\to\infty} n \ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{1/n} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1.$$

Then

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1.$$

Therefore, the series diverges by ratio test.

Theorem 10 (Root Test). *Consider* $\sum a_n$*, and suppose* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$*, then*

a) *If* $L < 1$*, then* $\sum a_n$ *converges absolutely,*
b) *If* $L > 1$*, then* $\sum a_n$ *diverges,*
c) *and if* $L = 1$*, the test is inconclusive.*

State whether the following converge or diverge, and state why

Ex: $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.

Solution: We apply the root test,

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left|\left(\frac{2n+3}{3n+2}\right)^n\right|} = \frac{2n+3}{3n+2} = \frac{2+3/n}{3+2/n}.$$

Taking the limit gives $\lim_{n\to\infty} \frac{2+3/n}{3+2/n} = \frac{2}{3} < 1$. Therefore, by the root test, the series converges absolutely.

Here are some problems from the book that we did in class:

19) We use ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\cancelto{n+1}{(n+1)!}(-e)^{-(n+1)}}{\cancel{n!}(-e)^{-n}}\right| = \left|(n+1)\frac{(-e)^n}{(-e)^{n+1}}\right| = \frac{1}{e}(n+1).$$

Taking the limit of this gives,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{e}(n+1) = \infty > 1$$

Therefore, the series diverges by ratio test.

29) Notice that

$$\frac{1}{2n} \leq \frac{1}{n} - \frac{1}{n^2}$$

And $\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges by p-test since $p = 1$, therefore the original series diverges by DCT.

35) We use ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(n+4)!}{3!(n+1)!3^{n+1}}}{\frac{(n+3)!}{3!n!3^n}}\right| = \left|\frac{\cancelto{n+4}{(n+4)!}}{\cancel{3!}\cancel{3^{n+1}}\cancelto{n+1}{(n+1)!}}\cdot\frac{\cancel{3!}\cancel{n!}\cancel{3^n}}{\cancel{(n+3)!}}\right| = \frac{n+4}{3n+3}$$

Taking the limit gives us

$$\lim_{n\to\infty}\frac{n+4}{3n+3} = \lim_{n\to\infty}\frac{1+4/n}{3+3/n} = \frac{1}{3} < 1.$$

Therefore, the series converges by ratio test.

57) Notice

$$\sum_{n=1}^{\infty}\frac{(n!)^n}{(n^n)^2} = \sum_{n=1}^{\infty}\left(\frac{n!}{n^2}\right)^n$$

We use root test,

$$\sqrt[n]{|a_n|} = \frac{n!}{n^2} = \frac{1\cdot 2\cdots(n-2)\cdot(n-1)\cdot n}{n^2} = 1\cdot 2\cdots(n-2)\cdot\left(1-\frac{1}{n}\right).$$

Taking the limit gives us

$$\lim_{n\to\infty}(n-2)!\left(1-\frac{1}{n}\right) = \infty > 1.$$

Therefore, the original series diverges by root test.