Math 2450 Rahman Exam III Sample Problems

(1) Evaluate the integral $\iint xydA$, over the region enclosed in the first quadrant, outside the circle $r = 1$ and inside the circle $r = 2\cos\theta$.

Solution:

$$\int_0^{\pi/2}\int_1^{2\cos\theta}(r\cos\theta)(r\sin\theta)rdrd\theta = \int_0^{\pi/2}\int_1^{2\cos\theta} r^3\cos\theta\sin\theta drd\theta = \int_0^{\pi/2}\left[\frac{1}{4}r^4\right]_1^{2\cos\theta}\cos\theta\sin\theta d\theta$$
$$= \int_0^{\pi/2}\frac{1}{4}\left[16\cos^4\theta - 1\right]\cos\theta\sin\theta d\theta = -\int_1^0\frac{1}{4}\left[16u^4-1\right]udu$$
$$= -\frac{1}{8}u^2 + \frac{2}{3}u^6\Big|_0^1 = \boxed{\frac{13}{24}}.$$

(2) Compute $\iint\limits_R (2x-3)dA$ where R is the region enclosed by the curves $y = x+4$ and $y = x^2-2x$.

Solution: Intersection: $x = -1$ and $x = 4$, then

$$\int_{-1}^4\int_{x^2-2x}^{x+4}(2x-3)dydx = \int_{-1}^4(2x-3)(4+3x-x^2)dx = \int_{-1}^4(-2x^3+9x^2-x-12)dx$$
$$= \left[-\frac{1}{2}x^4+3x^3+\frac{1}{2}x^2-12x\right]_{-1}^4 = \boxed{15}.$$

(3) Integrate

$$\int_{-1}^2\int_3^6(2x^2y-3x)dydx.$$

Solution:

$$\int_{-1}^2\int_3^6(2x^2y-3x)dydx = \int_{-1}^2\left[x^2y^2-3xy\right]_3^6dx = \int_{-1}^2\left[27x^2-18x-9x^2+9x\right]dx$$
$$= \int_{-1}^2\left[27x^2-9x\right]dx = 9x^3-\frac{9}{2}x^2\Big|_{-1}^2 = \boxed{\frac{135}{2}}.$$

(4) Reverse the order of integration to evaluate

$$\int_0^1\int_{3y}^3 e^{x^2}dxdy.$$

Solution: We first reverse the domain

$$D = \{(x,y)|0\le y\le 1,\ 3y\le x\le 3\} = \left\{(x,y)|0\le x\le 3,\ 0\le y\le\frac{x}{3}\right\}$$

Then

$$\int_0^3\int_0^{x/3}e^{x^2}dydx = \int_0^3[y]_0^{x/3}e^{x^2}dx = \frac{1}{3}\int_0^3 xe^{x^2}dx = \frac{1}{6}e^{x^2}\Big|_0^3 = \boxed{\frac{1}{6}\left(e^9-1\right)}.$$

(5) Using cylindrical coordinate find the volume of the region between the paraboloid $z = 9 - x^2 - y^2$, the plane $z = 0$, and the cylinder $x^2 + y^2 = 1$.

Solution:

$$\int_0^{2\pi}\int_0^1\int_0^{9-r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left(9 - r^2\right) r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left(9r - r^3\right) dr\,d\theta$$
$$= \int_0^{2\pi}\left[3r^2 - \frac{1}{4}r^4\right]_0^1 d\theta = \int_0^{2\pi}\frac{11}{4}d\theta = \boxed{\frac{11}{2}\pi}.$$

(6) Use cylindrical or polar coordinates to find the volume of the region bounded by $z = 2 - x^2 - y^2$ and $z = \sqrt{x^2 + y^2}$.

Solution: First we figure out where they intersect in cylindrical coordinates: $2 - r^2 = r \Rightarrow r = 1$. Notice that the intersection has no θ dependence. Also, $z = 2 - r^2$ is on top and $z = r$ is on the bottom

$$\int_0^{2\pi}\int_0^1 \left(-r^2 - r + 2\right) r\,dr\,d\theta = 2\pi\left[-\frac{1}{4}r^3 - \frac{1}{3}r^2 + r\right]_0^1 = \boxed{\frac{5}{6}\pi}.$$

(7) Find the area in the xy-plane bounded by $y = 0$, $x = 0$, $y = 1$, and $y = \ln x$.

Solution: Notice that there is only one boundary in x but three in y, however $y = \ln x \Rightarrow x = e^y$, so this must be the other boundary in x.

$$\int_0^1\int_0^{e^y} dx\,dy = \int_0^1 e^y dy = \boxed{e - 1}.$$

(8) Reverse the order and evaluate

$$\int_0^{\pi}\int_x^{\pi}\frac{\sin y}{y}dy\,dx$$

Solution: Again, lets reverse the domain first

$$D = \{(x, y)|0 \le x \le \pi,\ x \le y \le \pi\} = \{(x, y)|0 \le y \le \pi,\ 0 \le x \le y\}$$

then

$$\int_0^1\int_0^y \frac{\sin y}{y}dx\,dy = \int_0^1 \sin y\,dy = \boxed{\cos(1) - 1}.$$

(9) Use a triple integral to find the volume of the solid in the first octant that is bounded by $x = 0$, $y = 0$, $z = 0$, and

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Solution: Lets use x as our independent variable and go from there. If y and z are 0, then $0 \le x \le a$, next if $z = 0$, $0 \le y \le b - \frac{b}{a}x$, and finally $0 \le z \le c - \frac{c}{a}x - \frac{c}{b}y$. The integral becomes

$$\begin{aligned}
\int_0^a \int_0^{b-bx/a} \int_0^{c-cx/a-cy/b} dzdydx &= \int_0^a \int_0^{b-bx/a} \left(c - \frac{c}{a}x - \frac{c}{b}y\right) dydx = \int_0^a \left[cy - \frac{c}{a}xy - \frac{c}{2b}y^2\right]_0^{b-bx/a} \\
&= \int_0^a \left[c\left(b - \frac{b}{a}x\right) - \frac{c}{a}\left(bx - \frac{b}{a}x^2\right) - \frac{c}{2b}\left(b - \frac{b}{a}x\right)^2\right] \\
&= \left[cbx - \frac{cb}{a}x^2 + \frac{cb}{3a^2}x^3 + \frac{ca}{6b^2}\left(b - \frac{b}{a}x\right)^3\right]_0^a \\
&= \boxed{\frac{1}{3}cba}.
\end{aligned}$$

(10) Reverse the order and evaluate

$$\int_0^2 \int_y^2 e^{x^2} dxdy.$$

Solution: Once again,

$$D = \{(x, y) | 0 \le y \le 2,\ y \le x \le 2\} = \{(x, y) | 0 \le y \le 2,\ 0 \le y \le x\}$$

and

$$\int_0^2 \int_0^x e^{x^2} dydx = \int_0^2 xe^{x^2} dx = \frac{1}{2}e^{x^2}\Big|_0^2 = \boxed{\frac{1}{2}\left(e^4 - 1\right)}.$$

(11) Find the area of the region bounded by $x = y - y^2$ and $y = -x$.

Solution: Intersection: $-y = y - y^2 \Rightarrow y^2 - 2y = y(y - 2) = 0 \Rightarrow y = 0,\ 2$, then

$$\int_0^2 \int_{-y}^{y-y^2} dxdy = \int_0^2 \left[2y - y^2\right] dy = y^2 - \frac{1}{3}y^3\Big|_0^2 = \boxed{\frac{4}{3}}.$$