SECTIONS 9.1 - 9.4 (REVIEW)

We briefly went over a bunch of topics that were covered in Calc II.

Distance formula in 3-D: If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$,

$$d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

Know how to add vectors, multiply vectors by a scalar, find the magnitude of vectors, the different forms of writing a vector (e.g., column vector, using $\hat{\mathbf{i}} = (1, 0, 0), \hat{\mathbf{j}} = (0, 1, 0), \hat{\mathbf{k}} = (0, 0, 1)$), and how to find a unit vector.

We sketched some simple planes, such as $z = 3$, $y = 5$, and $y = x$.

Ex: The distance between the points $P = (2, -1, 7)$ and $Q = (1, -3, 5)$ is $d(P, Q) = 3$.

Equation of sphere: By definition a sphere is the set of all points $P = (x, y, z)$ whose distance from the center $C = (h, k, l)$ is r, so $d(P, C) = r \Rightarrow d(P, C)^2 = r^2 \Rightarrow (x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$.

Ex: Show that $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$ is the equation of a sphere.
Solution: Complete the square:

$$(x^2 + 4x + 4) + (y^2 - 6y + 9) + (z^2 + 2z + 1) = -6 + 4 + 9 + 1$$
$$\Rightarrow (x + 2)^2 + (y - 3)^2 + (z + 1)^2 = 8,$$

which means the center is $C = (2, -3, 1)$ and $r = 2\sqrt{2}$.

Ex: What region in $\mathbb{R}^3$ is represented by $1 \leq x^2 + y^2 + z^2 \leq 4$; $z \leq 0$?
Solution: In class we sketched this, but here we will forgo the sketch. Just recall that it is a hemisphere in the bottom $\mathbb{R}^3$ with a sphere of radius unity hollowed out.

<u>Properties of vectors</u>

(1) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$,
(2) $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$,
(3) $\vec{a} + \vec{0} = \vec{a}$,
(4) $\vec{a} + (-\vec{a})$,
(5) $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$,
(6) $(c + d)\vec{a} = c\vec{a} + d\vec{a}$,
(7) $(cd)\vec{a} = c(d\vec{a})$,
(8) $1\vec{a} = \vec{a}$

<u>Dot products</u> Recall that a dot product in $\mathbb{R}^2$ works as such

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix} = 2 \times 3 + 4 \times (-1) = 2.$$

In $\mathbb{R}^3$ it works the same way

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1b_1 + a_2b_2 + a_3b_3 \tag{1}$$

Properties of dot products

(1) $\vec{a} \cdot \vec{a} = \|a\|^2$,
(2) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$,
(3) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$,
(4) $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b})$,
(5) $\vec{0} \cdot \vec{a} = 0$.

Law of cosines: $\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta$

Ex: If $\|\vec{a}\| = 4$ and $\|\vec{b}\| = 6$, and $\theta = \pi/3$ (angle between $\vec{a}$ and $\vec{b}$), then $\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta = 12$.

Ex: Find the angle between $\vec{a} = \langle 1, 2, -1 \rangle$ and $\vec{b} = \langle 5, -3, 2 \rangle$.
Solution: $\|\vec{a}\| = 3$, $\|\vec{b}\| = \sqrt{38}$, $\vec{a} \cdot \vec{b} = 2$, then $\theta = \cos^{-1}\left(2/3\sqrt{38}\right)$.

Two nonzero vectors $\vec{a}$ and $\vec{b}$ are orthogonal (perpendicular) if and only if $\vec{a} \cdot \vec{b} = 0$. This is due to the law of cosines. When we have two vectors that are perpendicular the angle between them is $\pi/2$, and therefore $\cos\theta = 0$.

Ex: Show that $2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ is perpendicular to $5\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$.
Solution: $\langle 2, 2, -1 \rangle \cdot \langle 5, -4, 2 \rangle = 0$

Direction angles and cosines

Suppose α, β, γ are angles of a vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ from the x, y, z axes. Then

$$\cos\alpha = \frac{\vec{a} \cdot \hat{\mathbf{i}}}{\|\vec{a}\|\|\hat{\mathbf{i}}\|} = \frac{a_1}{\|\vec{a}\|}, \qquad \cos\beta = \frac{\vec{a} \cdot \hat{\mathbf{j}}}{\|\vec{a}\|\|\hat{\mathbf{j}}\|} = \frac{a_2}{\|\vec{a}\|}, \qquad \cos\gamma = \frac{\vec{a} \cdot \hat{\mathbf{k}}}{\|\vec{a}\|\|\hat{\mathbf{k}}\|} = \frac{a_3}{\|\vec{a}\|},$$

Projections

Suppose we want to find the projection of one vector $\vec{b}$ onto another vector $\vec{a}$. That is, find a vector in the direction of $\vec{a}$ with a magnitude that is equivalent to the magnitude of the component of $\vec{b}$ in the direction of $\vec{a}$.

The component of $\vec{b}$ in the direction of $\vec{a}$ will be

$$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|},$$

then the projection is simply

$$\text{proj}_{\vec{a}}\vec{b} = \left(\text{comp}_{\vec{a}}\vec{b}\right) \frac{\vec{a}}{\|\vec{a}\|} = \left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|}\right) \frac{\vec{a}}{\|\vec{a}\|}.$$

Ex: Project $\vec{b} = \langle 1, 1, 2 \rangle$ onto $\langle -2, 3, 1 \rangle$

$$\text{comp}_{\vec{a}}\vec{b} = \frac{3}{\sqrt{14}} \Rightarrow \text{proj}_{\vec{a}}\vec{b} = \frac{3}{\sqrt{14}} \frac{\vec{a}}{\|\vec{a}\|} = \begin{bmatrix} -3/7 \\ 9/14 \\ 3/14 \end{bmatrix}$$

Ex: Recall that work is a dot product of force and distance: $W = Fd\cos\theta = \vec{F} \cdot \vec{d}$.

Cross products

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then through cofactor expansion we get

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{\mathbf{k}} = (a_2 b_3 - a_3 b_2)\hat{\mathbf{i}} - (a_1 b_3 - a_3 b_1)\hat{\mathbf{j}} + (a_1 b_2 - a_2 b_1)\hat{\mathbf{k}}.$$

Ex: If $\vec{a} = \langle 1, 3, 4 \rangle$ and $\vec{b} = \langle 2, 7, -5 \rangle$, $\vec{a} \times \vec{b} = -43\hat{\mathbf{i}} + 13\hat{\mathbf{j}} + \hat{\mathbf{k}}$.

Ex: $\vec{a} \times \vec{a} = 0$.

Recall that the vector from the cross product $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$. This gives us the right hand rule that you would have learned in physics. And we can prove this by using the dot product: $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$.

Here are some other consequences from the cross product:

- If θ is the angle between $\vec{a}$ and $\vec{b}$, $\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin\theta$.
- Two vectors are parallel if and only if $\vec{a} \times \vec{b} = 0$.
- The length of $\vec{a} \times \vec{b}$ is given by the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.

Properties of cross products

(1) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ (i.e., does not commute),
(2) $(c\vec{a} \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$,
(3) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$,
(4) $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$,
(5) $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$,
(6) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$.

Another interesting consequence of the cross product is the volume of a parallelepiped. Suppose the vectors $\vec{a}, \vec{b}, \vec{c}$ make up the sides of the parallelepiped, then the magnitude of the triple product gives us the volume: $V = |\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)|$.

Ex: Find a vector perpendicular to the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

Solution: While we may not have the origin in this plane, we may redefine the "origin" to be a point on the plane, lets say P. Then we have two vectors: $\vec{PQ} = Q - P = \langle -3, 1, -7 \rangle$ and $\vec{PR} = P - R = \langle 0, -5, -5 \rangle$. Now the cross product gives us the perpendicular vector

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} = -40\hat{\mathbf{i}} - 15\hat{\mathbf{j}} + 15\hat{\mathbf{k}}$$

Ex: Lets do the following interesting derivation

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \hat{k}$$

$$\Rightarrow \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} a_1 - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} a_2 + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} a_3 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Physics/Engineering examples.

Ex: A crate is hauled $8m$ up a ramp under a constant force of $200N$ applied at an angle of $25°$ to the ramp. Find the work done.

Solution: We sketched a free-body diagram and notice that whether we use basic Trigonometry or law of cosines with the dot product we get the same thing:

$$W = F \cdot d = \|F\| \|d\| \cos(25°) = 1450J.$$

Ex: A force $F = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ moves an object from point $P(2, 1, 0)$ to point $Q(4, 6, 2)$. Find the work done.

Solution: $W = F \cdot d = F \cdot \vec{PQ} = 36.$

Recall that torque can be computed using the cross product,

$$\tau = r \times F \Rightarrow \|\tau\| = \|r \times F\| = \|r\| \|F\| \sin \theta.$$

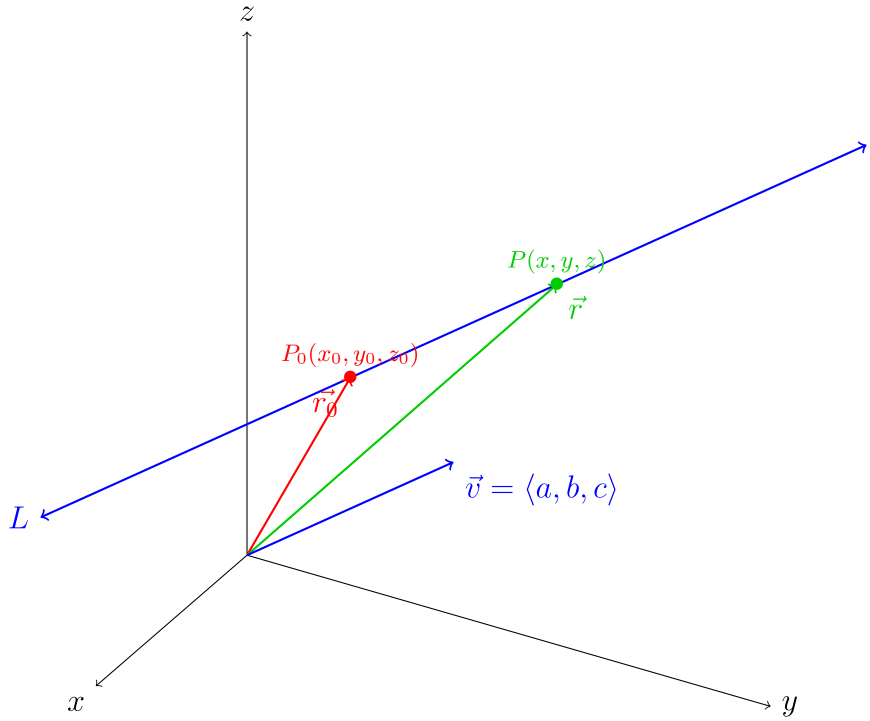
Ex: A bolt is tightened by applying a $40N$ force to a $0.25m$ wrench at a $75°$ angle. Find the magnitude of the torque.

Solution:

$$\|\tau\| = \|r \times F\| = \|r\| \|F\| \sin(75°) = (0.25)(40) \sin(75°) = 10 \sin(75°).$$

If the bolt is right-threaded it will go out of the board.

How do we determine a line in $\mathbb{R}^2$? We take a point on the line and find the direction (slope). The easiest point to use is the y-intercept: $y = mx + b$.



Similarly for $\mathbb{R}^3$ suppose we want to determine a line L that crosses through some point $P_0(x_0, y_0, z_0)$. We can't use the y-intercept and the slope because the line may not intersect one of the axes, and we don't have a slope (recall that slope is rise over run).

However, we can us a point on the line, say P_0 and find a direction, which is a vector $\vec{v}$ from the origin that is parallel to the line. In order to find this direction define an arbitrary point on the line: $P(x, y, z)$. Then $\vec{P_0P} = \langle P - P_0\rangle = \vec{r} - \vec{r}_0$ gives us the direction. Since they are in the same direction, $\vec{P_0P} = \vec{r} - \vec{v}_0 = t\vec{v}$ where the scalar t is the parameter of the line. Then the line is uniquely determined by $\vec{r} - \vec{r}_0 = t\vec{v}$. Since $\vec{r} = \langle x, y, z\rangle$ and $\vec{r}_0 = \langle x_0, y_0, z_0\rangle$, we can write down the vector equation of the line

$$\boxed{\langle x, y, z\rangle = \vec{r} = \langle x_0 + at, y_0 + bt, z_0 + ct\rangle}. \tag{2}$$

Some more algebra gives us the parametric representation of a line

$$\boxed{x = x_0 + at, \qquad y = y_0 + bt, \qquad z = z_0 + ct.} \tag{3}$$

As t varies we move up and down the line.