12.2 Double integration over general regions (continued)

Ex: $\iint\limits_D xy dA$ where D is bounded by $y = x - 1$ and $y^2 = 2x + 6$.

Again, we have two choices, but which is better?

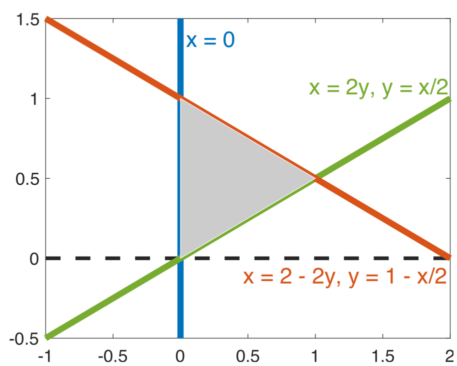
Notice that in the x-direction it is a pure function of y; i.e., we don't have any risk of having to split the interval.

Solution:

$$\int_{-2}^{4}\int_{y^2/2-3}^{y+1} xydxdy = \int_{-2}^{4}\left[\frac{x^2}{2}y\right]_{y^2/2-3}^{y+1} dy = \frac{1}{2}\int_{-2}^{4} y\left[(y+1)^2 - \left(\frac{1}{2}y^2 - 3\right)^2\right] dy$$

$$= \frac{1}{2}\int_{-2}^{4}\left(-\frac{1}{4}y^5 + 4y^3 + 2y^2 - 8y\right) dy = \frac{1}{2}\left[-\frac{y^6}{24} + y^4 + \frac{2}{3}y^3 - 4y^2\right]_{-2}^{4} = \boxed{36}.$$

Ex: Find the volume of the tetrahedron bounded by $x + 2y + z = 2$, $x = 2y$, $x = z = 0$.



Errata: in section 23 I used the triangle for $y = 0$ instead of $x = 0$, and in section 12 I should have used functions of x not y. Here is the correct version, and lets think about it in terms vertical or horizontal cross-sections.

If we sketch it we notice that we get a natural upper and lower limit for the boundary if we take vertical slices; whereas if we take horizontal slices we need to split the integral. I will show both ways here.

$$V = \int_0^1\int_{x/2}^{1-x/2}(2 - x - 2y)dydx = \int_0^1 [2y - xy - y^2]_{x/2}^{1-x/2} dx$$

$$= \int_0^1\left[2 - x - x\left(1 - \frac{x}{2}\right) - \left(1 - \frac{x}{2}\right)^2 - x\frac{x^2}{2} + \frac{x^2}{4}\right] dx$$

$$= \int_0^1 \left(x^2 - 2x + 1\right) dx = \frac{x^3}{3} - x^2 + x\Big|_0^1 = \boxed{\frac{1}{3}}.$$

and

$$
\begin{aligned}
V &= \int_0^{1/2}\int_0^{2y}(2-x-2y)dxdy + \int_{1/2}^{1}\int_0^{2-2y}(2-x-2y)dxdy \\
&= \int_0^{1/2}\left[2x-\frac{1}{2}x^2-2xy\right]_0^{2y}dy + \int_{1/2}^{1}\left[2x-\frac{1}{2}x^2-2xy\right]_0^{2-2y}dy \\
&= \int_0^{1/2}\left[4y-\frac{1}{2}(4y^2)-2(2y)y\right]dy + \int_{1/2}^{1}\left[2(2-2y)-\frac{1}{2}(2-2y)^2-2(2-2y)y\right]dy \\
&= \int_0^{1/2}\left[4y-6y^2\right]dy + \int_{1/2}^{1}\left[2y^2-4y+2\right]dy = \left[2y^2-2y^3\right]_0^{1/2} + \left[\frac{2}{3}y^3-2y^2+2y\right]_{1/2}^{1} = \boxed{\frac{1}{3}}
\end{aligned}
$$

We notice that the first way is much easier.

Ex: $\int_0^1\int_x^1 \sin(y^2)dydx$

Solution: Notice that this is a nasty integral. Lets reverse it by writing the domain of the integral and then writing the equivalent domain if we reverse it.

$$D = \{(x,y)|0 \le x \le 1,\, x \le y \le 1\} = \{(x,y)|0 \le y \le 1, 0 \le x \le y\}.$$

$$
\begin{aligned}
\int_0^1\int_x^1 \sin(y^2)dydx &= \int_0^1\int_0^y \sin(y^2)dxdy = \int_0^1 \left[x\sin(y^2)\right]_0^y dy \\
&= \int_0^1 y\sin(y^2)dy = -\frac{1}{2}\cos(y^2)\Big|_0^1 = \frac{1}{2}(1-\cos(1)).
\end{aligned}
$$

12.3 Double integrals in polar coordinates

Recall $r^2 = x^2+y^2$, $x = r\cos\theta$, and $y = r\sin\theta$. In class we derived the area element $dA = rdrd\theta$, which comes from the length of one arc of a circle. What we need to remember is If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, and $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x,y)dA = \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)rdrd\theta. \tag{1}$$

Ex: $\iint \times_R (3x+4y^2)dA$ where R is bounded by $x^2+y^2=1$ and $x^2+y^2=4$ in the upper xy-plane.

Solution: Our region will be

$$R = \{(x,y)|y \geq 0, 1 \leq x^2+y^2 \leq 4\} = \{(r,\theta)|1 \leq r \leq 2, 0 \leq \theta \leq \pi\}.$$

then the integral is

$$\int_0^\pi \int_1^2 \left(3r\cos\theta + 4r^2\sin^2\theta\right) r dr d\theta = \int_1^\pi \int_1^2 \left(3r^2\cos\theta + 4r^3\sin^2\theta\right) dr d\theta = \int_0^\pi \left[r^3\cos\theta + r^4\sin^2\theta\right]_{r=1}^{r=2} d\theta$$

$$= \int_0^\pi \left(7\cos\theta + 15\sin^2\theta\right) d\theta = \int_0^\pi \left[7\cos\theta + \frac{15}{2}(1-\cos 2\theta)\right] d\theta$$

$$= 7\sin\theta + \frac{15}{2}\theta - \frac{15}{4}\sin 2\theta\Big|_0^\pi = \boxed{\frac{15}{2}\pi}$$

Ex: Find the volume of a solid bounded by $z=0$ and $z=1-x^2-y^2$.

Solution:

$$V = \int_0^{2\pi} \int_0^1 (1-r^2) r dr d\theta = \int_0^{2\pi} d\theta \int_0^1 (r-r^3)dr = 2\pi\left[\frac{1}{2}r^2 - \frac{1}{4}r^4\right]_0^1 = \boxed{\frac{\pi}{2}}.$$

Just like in rectangular, we can have boundaries that are functions. If f is continuous on

$$D = \{(r,\theta)|\alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\} \tag{2}$$

then

$$\iint_D f(x,y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r dr d\theta. \tag{3}$$

Ex: Find the area enclosed by one loop of $r=\cos\theta$.

We could do this as a single integral, but lets see how this would work as a double integral.

Solution: To get one loop we would need two consecutive angles where $r=0$.

$$D = \left\{(r,\theta)| -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4},\ 0 \leq r \leq \cos 2\theta\right\}$$

then the integral becomes

$$\iint_D dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r dr d\theta = \int_{-\pi/4}^{\pi/4} \left[\frac{1}{2}r^2\right]_0^{\cos 2\theta} d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta d\theta$$

$$= \frac{1}{4}\int_{-\pi/4}^{\pi/4} (1+\cos 4\theta)d\theta = \frac{1}{4}\left[\theta + \frac{1}{4}\sin 4\theta\right]_{-\pi/4}^{\pi/4} = \boxed{\frac{\pi}{8}}.$$

Ex: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

Solution: Lets convert everything into polar first: $f(r, \theta) = r^2$, $x^2 + y^2 = 2x \Rightarrow r^2 = 2r\cos\theta$ at the boundary. Then

$$D = \{(r, \theta)| -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},\ 0 \leq r \leq 2\cos\theta\}.$$

Then the integral becomes

$$V = \iint_D (x^2 + y^2)dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^3 dr d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{4} r^4 \Big|_0^{2\cos\theta} d\theta = 4 \int_{-\pi/2}^{\pi/2} \cos^4\theta d\theta$$

$$= 8 \int_0^{\pi/2} \cos^4\theta d\theta = 8 \int_0^{\pi/2} \left[\frac{1}{2}(1 + \cos 2\theta)\right]^2 d\theta = 2 \int_0^{\pi/2} \left[1 + 2\cos 2\theta + \frac{1}{2}(1 + \cos 4\theta)\right] d\theta$$

$$= 2\left[\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta\right]_0^{\pi/2} = \boxed{\frac{3}{2}\pi}.$$

12.4 Surface Area

The surface area is derived similarly to arc length, and therefore have similar formulas. If $z = f(x, y)$,

$$SA = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA. \quad (4)$$

Ex: Find the surface area of $z = x^2 + 2y$ that lies above the triangular region T in the xy-plane with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$.

Solution: The region can be written as

$$T = \{(x, y)|0 \leq x \leq 1,\ 0 \leq y \leq x\}.$$

Then

$$SA = \int_0^1 \int_0^x \sqrt{4x^2 + 5}\, dy dx = \int_0^1 x\sqrt{4x^2 + 5}\, dx = \frac{1}{8} \cdot \frac{2}{3} \left(4x^2 + 5\right)^{3/2} \Big|_0^1 = \boxed{\frac{1}{12}\left(27 - 5\sqrt{5}\right)}.$$

Ex: Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

Solution: The boundary of intersection will be $x^2 + y^2 = 9 \Rightarrow r^2 = 9$, and $\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{4x^2 + 4y^2 + 1} = \sqrt{4r^2 + 1}$, then

$$SA + \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}\, r dr d\theta = \int_0^{2\pi} d\theta \int_0^3 \sqrt{1 + 4r^2}\, r dr = 2\pi \left[\frac{1}{8} \cdot \frac{2}{3} \left(1 + 4r^2\right)^{3/2}\right]_0^3 = \boxed{\frac{\pi}{6}(37\sqrt{37} - 1)}.$$

This works just like double and single integrals, so lets skip the B.S. and get straight to doing problems.

Ex: Evaluate the triple integral $\iiint\limits_B xyz^2 dV$ where B is the rectangular box

$$B = \{(x,y,z) | 0 \le x \le 1, -1 \le y \le 2, 0 \le z \le 3\}$$

Solution: We simply integrate to get

$$\iiint\limits_B xyz^2 dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2 dxdydz = \int_0^3 \int_{-1}^2 \left[\frac{1}{2}x^2\right]_0^1 yz^2 dydz = \int_0^3 \int_{-1}^2 \frac{yz^2}{2} dydz$$

$$= \int_0^3 \left[\frac{1}{4}y^2\right]_{-1}^2 z^2 dz = \int_0^3 \frac{3}{4} z^2 dz = \left.\frac{z^3}{4}\right|_0^3 = \boxed{\frac{27}{4}}.$$

Ex: Evaluate $\iiint\limits_E zdV$, where E is the solid tetrahedron bounded by the four planes $x = y = z = 0$ and $x + y + z = 1$.

Solution: For this one it may be a good idea to sketch it first. We let $y = z = 0$ to get the bounds for x, and let $z = 0$ to get the bounds for y,

$$E = \{(x,y,z) | 0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le 1 - x - y\}.$$

Then

$$\iiint\limits_E zdV = \int_0^1 \int 0^{1-x} \int_0^{1-x-y} zdzdydx = \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2}\right]_0^{1-x-y} dydx = \frac{1}{2}\int_0^1 \int_0^{1-x} (1-x-y)^2 dydx$$

$$= \frac{1}{2}\int_0^1 \left[-\frac{1}{3}(1-x-y)^3\right]_0^{1-x} dx = \frac{1}{6}\int_0^1 (1-x)^3 dx = \frac{1}{6}\left[-\frac{1}{4}(1-x)^4\right]_0^1 = \boxed{\frac{1}{24}}.$$

Ex: Evaluate $\iiint\limits_E \sqrt{x^2+z^2} dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and $y = 4$.

Solution: Notice that the y limits are already given and the Kernel of the integral doesn't have y in it, so lets do that integral first.

$$\iiint\limits_E \sqrt{x^2+z^2} dV = \iint\limits_D \left[\int_{x^2+z^2}^4 \sqrt{x^2+z^2} dy\right] dA = \iint\limits_D (4 - x^2 - z^2)\sqrt{x^2+z^2} dA.$$

What are some ideas to make that integral easier?

$$\iint\limits_D (4-x^2-z^2)\sqrt{x^2+z^2} dA = \int_0^{2\pi} \int_0^2 (4-r^2)\sqrt{r^2} r dr d\theta = \int_0^{2\pi} \theta \int_0^2 \left(4r^2 - r^4\right) dr = 2\pi \left[\frac{4}{3}r^3 - \frac{1}{5}r^5\right]_0^2 = \boxed{\frac{128\pi}{15}}.$$