12.5 Triple integrals (continued)

Ex: Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

Solution: This one was quite a bit involved. But basically we would want to use x as the independent variable, and find out where two sides of the tetrahedron intersect. On the xy-plane we have $x = 2y$ and $x = 2 - 2y$ since $z = 0$, then they intersect at $x = 1$. So, x goes from 0 to 1, and we take horizontal slices for y on the xy-plane. Then our domain becomes

$$T = \{(x, y, z)|0 \leq x \leq 1,\ \frac{x}{2} \leq y \leq 1 - \frac{x}{2},\ 0 \leq z \leq 2 - x - 2y\}$$

Since we are finding a volume our Kernel of integration will be 1. Then

$$V = \iiint_T dV = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dzdydx = \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)dydx$$

$$= \int_0^1 [2y - xy - y^2]_{x/2}^{1-x/2} dx = \int_0^1 \left[2 - x - x\left(1 - \frac{x}{2}\right) - \left(1 - \frac{x}{2}\right)^2 - x\frac{x^2}{2} + \frac{x^2}{4}\right] dx$$

$$= \int_0^1 \left(x^2 - 2x + 1\right) dx = \frac{x^3}{3} - x^2 + x\Big|_0^1 = \boxed{\frac{1}{3}}.$$

12.7 Cylindrical and spherical coordinates

Cylindrical coordinates. Cylindrical coordinates are just like polar, except with a z component. For this class we will use the following format to do all cylindrical integrals:

$$\iiint_E f(x, y, z)dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta), r\sin\theta}^{u_2(r\cos\theta, r\sin\theta)} rdzdrd\theta. \tag{1}$$

Ex: Evaluate

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)dzdydx.$$

Solution: We need to convert our original limits to cylindrical. Notice that the x limits $-2 \leq x \leq 2$ and the y limits $-\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}$ give us the circle $x^2 + y^2 = 4$. Then

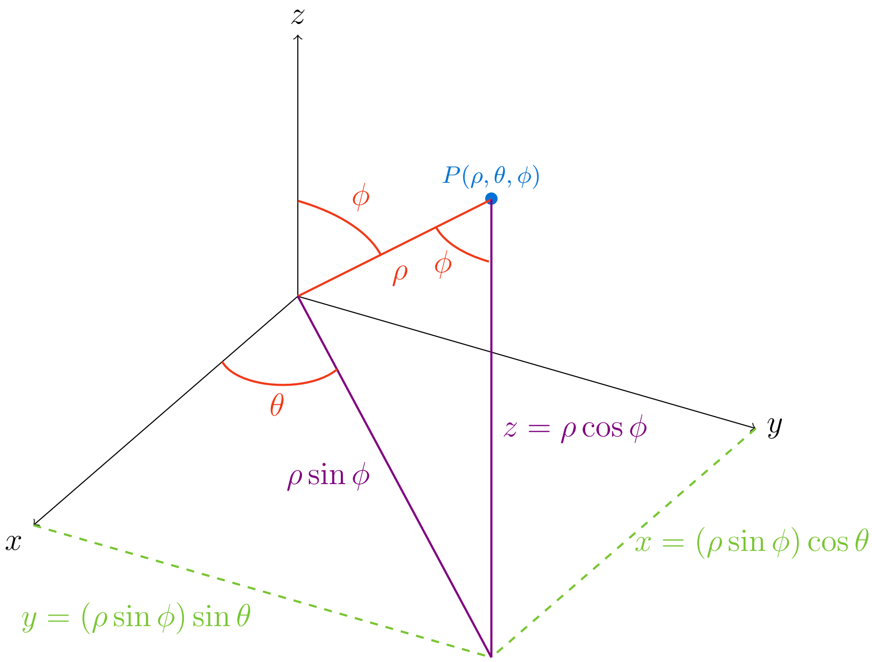
$$E = \{(x, y, z)| - 2 \leq x \leq 2,\ -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2},\ \sqrt{x^2 + y^2} \leq z \leq 2$$
$$= \{(r, \theta, z)|0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 2,\ r \leq z \leq 2\}.$$

And our integral becomes

$$\iiint_E (x^2 + y^2)dV = \int_0^{2\pi} \int_0^2 \int_r^2 (r^2)rdzdrd\theta = \int_0^{2\pi} \int_0^2 r^3[z]_r^2 drd\theta = \int_0^{2\pi} \int_0^2 r^3[2 - r]drd\theta$$

$$= 2\pi \int_0^2 (2r^3 - r^4)dr = 2\pi \left[\frac{1}{2}r^4 - \frac{1}{5}r^5\right]_0^2 = 2\pi \left[8 - \frac{32}{5}\right] = \boxed{\frac{16}{5}\pi}.$$

Spherical coordinates. Spherical on the other hand is completely different.



Our spherical integrals will always be in the following format:

$$\int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi d\rho d\theta d\phi. \tag{2}$$

Ex: Evaluate

$$\iiint_B e^{(x^2+y^2+z^2)^{3/2}} dV,$$

where

$$B = \{(x, y, z) | x^2 + y^2 + z^2 \leq 1\}.$$

Solution: Lets convert B,

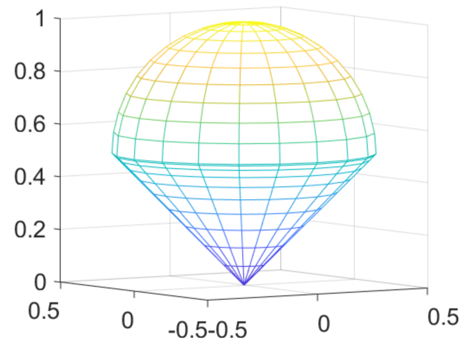
$$B = \{(\rho, \theta, \phi) | 0 \leq \rho \leq 1, \, 0 \leq \theta \leq 2\pi, \, 0 \leq \phi \leq \pi\}.$$

Then

$$\iiint_B e^{(x^2+y^2+z^2)^{3/2}} dV = \int_0^\pi \int_0^{2\pi} \int_0^1 e^{(\rho^2)^{3/2}} \rho^2 \sin \phi d\rho d\theta d\phi = 2\pi \int_0^\pi \sin \phi d\phi \int_0^1 e^{\rho^3} \rho^2 d\rho$$

$$= 2\pi [-\cos \phi]_0^\pi \left[\frac{1}{3} e^{\rho^3}\right]_0^1 = \boxed{\frac{4}{3}\pi(e-1)}.$$

Ex: Use spherical coordinates to find the volume of the solid above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = z$.



Solution: Notice that the intersection does not restrict θ, so $0 \leq \theta \leq 2\pi$. Further, the radius will go from the origin to the boundary of the sphere:

$$\rho^2 = \rho\cos\phi \Rightarrow 0 \leq \rho \leq \cos\phi.$$

Finally, we have ϕ. This will go from the z-axis to the cone, so we need to know the angle of the side of the cone:

$$z = \sqrt{x^2+y^2} \Rightarrow \rho\cos\phi = \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta} = \rho\sin\phi \Rightarrow \sin\phi = \cos\phi \Rightarrow \phi = \frac{\pi}{4},$$

then $0 \leq \phi \leq \pi/4$. Our integral becomes

$$V = \int_0^{\pi/4}\int_0^{2\pi}\int_0^{\cos\phi} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta = 2\pi\int_0^{\pi/4}\sin\phi\left[\frac{1}{3}\rho^3\right]_0^{\cos\phi} d\phi$$

$$= \frac{2\pi}{3}\int_0^{\pi/4}\sin\phi\cos^3\phi\, d\phi = \frac{2\pi}{3}\left[-\frac{1}{4}\cos^4\phi\right]_0^{\pi/4} = \boxed{\frac{\pi}{8}}.$$

12.8 The Jacobian

There is actually an algorithmic way to change the variables, and that is through the Jacobian.

If $x = g(u,v)$ and $y = h(u,v)$ then

$$J(x,y) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \tag{3}$$

And $dxdy = dA = J(x,y)dudv$. Similarly, if $x = g(u,v,w)$, $y = h(u,v,w)$, and $z = k(u,v,w)$, then

$$J(x,y,z) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}. \tag{4}$$