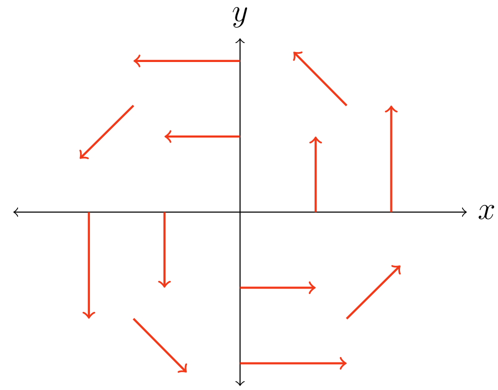
13.1 Vector fields

Let E be a subset of $\mathbb{R}^3$. A vector field on $\mathbb{R}^3$ is a vector function $\vec{F}$ that assigns a vector $\vec{F}(x,y,z)$ at each point (x,y,z). Note: this can be done in any number of dimensions, but we will stick to 2 and 3-D. Also we can write $\vec{F}$ as

$$\vec{F}(x,y,z) = P(x,y,z)\hat{\mathbf{i}} + Q(x,y,z)\hat{\mathbf{j}} + R(x,y,z)\hat{\mathbf{k}}. \tag{1}$$

Ex: A vector field on $\mathbb{R}^2$ is defined by $F(x,y) = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$. This creates the vectors in the figure on the right at each point in $\mathbb{R}^2$. Some representative points are taken in the figure on the right. Notice that from the representative points we can deduce that a particle under the influence of that vector field will go in a circle with the radius determined by the initial condition. Other examples of vector fields are fluid flows, gravity electricity, magnetism, etc. The gradient is also a type of vector field because $\nabla f = f_x\hat{\mathbf{i}} + f_y\hat{\mathbf{j}}$.



Curl

$$\mathbf{curl}(F) = \nabla \times F = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}. \tag{2}$$

Ex: If $F(x,y,z) = xz\hat{\mathbf{i}} + xyz\hat{\mathbf{j}} - y^2\hat{\mathbf{k}}$, find $\mathbf{curl}(F)$.

Solution:

$$\nabla \times F = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} = \boxed{-y(2+x)\hat{\mathbf{i}} + x\hat{\mathbf{j}} + yz\hat{\mathbf{k}}}.$$

Theorem 1. *If f is a function of three variables that has continuous second order partial derivatives, then*

$$\mathbf{curl}(\nabla f) = 0. \tag{3}$$

Proof.

$$\nabla \times (\nabla f) = (f_{yz} - f_{zy})\hat{\mathbf{i}} + (f_{zx} - f_{xz})\hat{\mathbf{j}} + (f_{xy} - f_{yx})\hat{\mathbf{k}} = 0.$$

□

Here ∇f is called a conservative vector field

Divergence

$$\mathbf{div}(F) = \nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}. \tag{4}$$

Ex: If $F(x,y,z) = xz\hat{\mathbf{i}} + xyz\hat{\mathbf{j}} - y^2\hat{\mathbf{k}}$, find $\mathbf{div}(F)$

Solution:

$$\mathbf{div}(F) = \nabla \cdot F = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2) = \boxed{z + xz}.$$

Theorem 2. *If $F = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second partial derivatives then*

$$\textbf{div}(\textbf{curl}(F)) = 0. \tag{5}$$

Proof.

$$\nabla \cdot (\nabla \times F) = \frac{\partial}{\partial x}(R_y - Q_z) + \frac{\partial}{\partial y}(P_z - R_x) + \frac{\partial}{\partial z}(Q_x - P_y) = 0.$$

□

The Laplacian

If the gradient is a generic first order derivative in multivariate systems, the Laplacian is the generic second order derivative.

$$\nabla^2 f = \nabla \cdot (\nabla f) = f_{xx} + f_{yy} + f_{zz}. \tag{6}$$

A function that satisfies Laplace's equation ($\nabla^2 f = 0$) is said to be Harmonic.

13.2 Line integrals

Suppose we have a curve, C, defined parametrically as $x = x(t)$, $y = y(t)$; $a \leq t \leq b$, and we want to integrate a function over this line. Then we must do a change of variables with the parameterization

Definition 1. If f is defined on a smooth curve, C, then the line integral of f along C is

$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt \tag{7}$$

While this is written in 2-D it can easily be extended to 3-D.

Ex: Evaluate $\int_C (2 + x^2 y)ds$ where C is $x^2 + y^2 = 1$; $y \geq 0$.

Solution: What is the obvious parameterization? $x = \cos t$, $y = \sin t$.

$$\int_C (2 + x^2 y)ds = \int_0^\pi (2 + \cos^2 t \sin t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt = \int_0^\pi (2 + \cos^2 t \sin t)\underbrace{\sqrt{\sin^2 t + \cos^2 t}}_{1}dt$$

$$= \int_0^\pi (2 + \cos^2 t \sin t)dt = \boxed{2\pi + \frac{2}{3}}.$$

Ex: Evaluate $\int_C 2x\,ds$ where C consists of the arc, C_1, of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment, C_2, from $(1,1)$ to $(1,2)$.

C_1: Since this is a function of x, we can parameterize as $x = x$, $y = x^2$; $0 \leq x \leq 1$, then

$$\int_{C_1} 2x\,ds = \int_0^1 2x\sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 2x\sqrt{1+4x^2}\,dx = \boxed{\frac{5\sqrt{5}-1}{6}}.$$

C_2: Notice that his is purely a function of y as x doesn't change, so let $x = 1$, $y = y$; $1 \leq y \leq 2$, then

$$\int_{C_2} 2x\,ds = \int_1^2 2 \cdot 1\sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2}\,dy = \int_1^2 2\,dy = \boxed{2}.$$

$C_1 + C_2$: Thus,

$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds = \boxed{\frac{5\sqrt{5}-1}{6} + 2}.$$