## 13.2 LINE INTEGRALS (CONTINUED)

Line integrals with respect to $x$ and $y$

If we want to integrate over a curve, but only take the $x$ or $y$ contributions, our change of variables becomes much easier, and hence the formulas are more compact.

$$\begin{aligned}\int_C f(x,y)dx &= \int_a^b f(x(t),y(t))x'(t)dt,\\ \int_C f(x,y)dy &= \int_a^b f(x(t),y(t))y'(t)dt.\end{aligned} \tag{1}$$

Ex: Evaluate $\int_C y^2dx + xdy$, where **a)** $C = C_1$ is the line segment from $(-5,-3)$ to $(0,2)$ and **b)** $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.

(a) In class we used a slightly more obvious parameterization, but here I am going to use something that looks different, but is in fact equivalent.
Parameterization: $x = 5t - 5$, $y = 5t - 3$; $0 \le t \le 1$. Then

$$\int_{C_1} y^2dx + xdy = \int_0^1 (5t-3)^2(5dt) + (5t-5)(5dt) = 5\int_0^1 (25t^2 - 25t + 4)dt$$
$$= 5\left[\frac{25t^3}{3} - \frac{25t^2}{2} + 4t\right]_0^1 = \boxed{-\frac{5}{6}}.$$

(b) Parameterization: $x = 4 - y^2$, $y = y$; $-3 \le y \le 2$.

$$\int_{C_2} y^2dx + xdy = \int_{-3}^2 y^2(-2y)dy + 4(4-y^2)dy = \int_{-3}^2 (-2y^3 - y^2 + 4)dy$$
$$= \left[-\frac{y^4}{2} - \frac{y^3}{3} + 4y\right]_{-3}^2 = \boxed{-\frac{245}{6}}.$$

Ex: Evaluate $\int_C y\sin z\,ds$ where $C$ is the circular helix $x = \cos t$, $y = \sin t$, $z = t$; $0 \le t \le 2\pi$.
**Solution:**

$$\int_C y\sin z\,ds = \int_0^{2\pi} (\sin t)\sin t\sqrt{\sin^2 t + \cos^2 t + 1}dt = \sqrt{2}\int_0^{2\pi} \frac{1}{2}(1 - \cos 2t)dt$$


$$= \frac{\sqrt{2}}{2}\left[t - \frac{1}{2}\sin 2t\right]_0^{2\pi} = \boxed{\sqrt{2}\pi}.$$

Ex: Evaluate $\int_C ydx + zdy + xdz$ where $C$ consists of line segments $C_1$ from $(2,0,0)$ to $(3,4,5)$ and $C_2$ from $(3,4,5)$ to $(3,4,0)$.
**Solution:**

$C_1$: We can write the parameterization in the form of a vector in 3-D: $\vec{r}(t) = \langle 2+t, 4t, 5t\rangle$. Then

$$\int_{C_1} ydx + zdy + xdz = \int_0^1 (4t)dt + (5t)(4dt) + (2+t)(5dt) = 10t + \frac{29}{2}t^2\Big|_0^1 = \boxed{\frac{49}{2}}.$$

$C_2$: Once again our vector is $\vec{r}(t) = \langle 3, 4, 5-5t\rangle$. Notice that $dx = dy = 0$, so

$$\int_{C_2} ydx + zdy + xdz = \int_0^1 3(-5)dt = -15.$$

$C_1 + C_2$: $\int_C ydx + zdy + xdz = 24.5 - 15 = 9.5$.

Line integrals of vector fields

If we want the integral along $C$ of $\vec{F}$ we need to pick out the component of $\vec{F}$ tangential to the curve; i.e., $\vec{F}\cdot\vec{T}$, and notice that $ds/dt = \|\vec{r}\,'(t)\|$, and $T = \vec{r}\,'(t)/\|\vec{r}\,'(t)\|$, so

$$\int_C F\cdot T ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}\|\vec{r}\,'(t)\|dt = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)dt. \tag{2}$$

Ex: Find the work done by the force field $F(x,y) = x^2\hat{\mathbf{i}} - xy\hat{\mathbf{j}}$ in moving a particle along the quarter-circle.

**Solution:** Parameterization: $\vec{r}(t) = \cos t\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}}$; $0 \le t \le \pi/2$.

So, $F(r(t)) = \cos^2 t\hat{\mathbf{i}} - \cos t\sin t\hat{\mathbf{j}}$ and $r'(t) = -\sin t\hat{\mathbf{i}} + \cos t\hat{\mathbf{j}}$. Therefore,

$$W = \int_C F\cdot dr = \int_0^{\pi/2} F(r(t))\cdot r'(t)dt = \int_0^{\pi/2}(-2\cos^2 t + \sin t)dt = \frac{2}{3}\cos^3 t\Big|_0^{\pi/2} = \boxed{-\frac{2}{3}}.$$

Ex: Evaluate $\int_C F\cdot dr$ where $F(x,y,z) = xy\hat{\mathbf{i}} + yz\hat{\mathbf{j}} + zx\hat{\mathbf{k}}$ and $C$ is the twisted cubic $x = t$, $y = t^2$, $z = t^3$; $0 \le t \le 1$.

**Solution:** $r(t) = \langle t, t^2, t^3\rangle \Rightarrow r'(t) = \langle 1, 2t, 3t^2\rangle$ and $F = \langle t^3, t^5, t^4\rangle$. Then

$$\int_C F\cdot dr = \int_0^1 F(r(t))\cdot r'(t)dt = \int_0^1 \left(t^3 + 5t^6\right)dt = \frac{1}{4}t^4 + \frac{5}{7}t^7\Big|_0^1 = \boxed{\frac{27}{28}}.$$

## 13.3 Fundamental theorem of line integrals

**Theorem 1.** *Let $C$ be a smooth curve given by the vector function $\vec{r}(t)$; $a \le t \le b$. Let $f$ be a differentiable function of two or three variables whose gradient vector $\nabla f$ is continuous on $C$. Then*

$$\int_C \nabla f\cdot dr = f(r(b)) - f(r(a)). \tag{3}$$

*Proof.*

$$\int_C \nabla f\cdot dr = \int_a^b \nabla f\cdot r'(t)dt = \int_a^b\left(f_x\frac{dx}{dt} + f_y\frac{dy}{dt} + f_z\frac{dz}{dt}\right)dt$$
$$= \int_a^b \frac{d}{dt}f(r(t))dt = f(r(b)) - f(r(a))$$

by the fundamental theorem of calculus. □

This basically says that we can evaluate the line integral of a conservative vector field $\vec{F} = \nabla f$ from the value of $f$ at the end points.

Ex: Find the work done by the gravitational field $F(x) = -mMG\vec{r}/\|\vec{r}\|^3$ in a moving particle with mass $m$ from point $(3,4,12)$ to point $(2,2,0)$ along a smooth curve.

**Solution:** This is a conservative vector field since $\nabla\times F = 0$, and therefore there is an $f$ such that $F = \nabla f$. This $f$ turns out to be $f(x,y,z) = mMG/\|\vec{r}\|$. Then

$$W = \int_C F\cdot dr = \int_C \nabla f\cdot dr = f(2,2,0) - f(3,4,12) = \frac{mMG}{\sqrt{2^2+2^2+0^2}} - \frac{mMG}{\sqrt{3^2+4^2+12^2}} = \boxed{mMG\left(\frac{1}{2\sqrt{2}} - \frac{1}{13}\right)}.$$

Properties of path independence

- $\int_C F \cdot dr$ is path independent if and only if $\int_C F \cdot dr = 0$ for all closed $C \subset D$.
- If $\int_C F \cdot dr$ is path independent, then $F$ is conservative.
- $F = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}}$ is conservative if and only if $P_y = Q_x$.

Ex: Determine whether $F(x, y) = (x - y)\hat{\mathbf{i}} + (x - 2)\hat{\mathbf{j}}$ is conservative. **Solution:** $P_y = -1$ and $Q_x = 1$, so $F$ is not conservative since $P_y \neq Q_x$.

Ex: Determine if $F(x, y) = (3 + 2xy)\hat{\mathbf{i}} + (x^2 - 3y^2)\hat{\mathbf{j}}$ is conservative. **Solution:** $P_y = 2x = Q_x$, so $F$ is conservative.

Ex: (a) If $F(x, y) = (3 + 2xy)\hat{\mathbf{i}} + (x^2 - 3y^2)\hat{\mathbf{j}}$, find $f$ such that $F = \nabla f$.
**Solution:** Since $\nabla f = \langle f_x, f_y\rangle$, $\langle f_x, f_y\rangle = \langle 3 + 2xy, x^2 - 3y^2\rangle$. So, $f_x = 3 + 2xy \Rightarrow f(x, y) = 3x + x^2y + g(y)$ because we need a constant of integration, but since $f$ is a function of two variables and $f_x$ will differentiate out any function of only $y$ we need our "constant" to be some generic function of $y$. Now we have an explicit form of $f_y$ from $\nabla f = F$, and we can differentiate the $f$ we found. If we equate them we will find what $g(y)$ is. $f_y = x^2 + g'(y) = x^2 - 3y^2$, so clearly $g'(y) = -3y^2 \Rightarrow g(y) = -y^3 + K$. Plugging this back gives us $f(x, y) = 3x + x^2y - y^3 + K$.

(b) Evaluate the line integral $\int_C F \cdot dr$ where $C$ is the curve given by $\vec{r}(t) = e^t \sin t\hat{\mathbf{i}} + e^t \cos t\hat{\mathbf{j}}$; $0 \leq t \leq \pi$.
**Solution:** $\vec{r}(0) = \langle 0, 1\rangle$ and $\vec{r}(\pi) = \langle 0, -e^\pi\rangle$, then

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(0, -e^\pi) - f(0, 1) = e^{3\pi} - (-1)e^{3\pi} + 1.$$

Ex: If $F(x, y, z) = y^2\hat{\mathbf{i}} + (2xy + e^{3z})\hat{\mathbf{j}} + 3ye^{3z}\hat{\mathbf{k}}$ find $f$ such that $\nabla f = F$.
**Solution:** $f_x = y^2 \Rightarrow f = xy^2 + g(y, z)$, then $f_y = 2xy + g_y = 2xy + e^{3z}$. Clearly, $g_y = e^{3z} \Rightarrow g = ye^{3z} + h(z)$. Again we plug back in and differentiate with $z$ this time $f = xy^2 + ye^{3z} + h(z) \Rightarrow f_z = 3ye^{3z} + h'(z) = 3ye^{3z}$. Clearly, $h'(z) = 0 \Rightarrow h(z) = K$, and finally $f = xy^2 + ye^{3z} + K$.

## 13.4 Green's theorem

**Theorem 2.** *Let $C$ be a positively oriented, piecewise-smooth, simple closed curve in the plane and let $D$ be the region bounded by $C$. If $P$ and $Q$ have continuous partial derivatives on an open region that contains $D$, then*

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA. \tag{4}$$

(1) Evaluate $\int_C x^4 dx + xy dy$ where $C$ is the triangular curve consisting of the line segments from $(0, 0)$ to $(1, 0)$, from $(1, 0)$ to $(0, 1)$, and from $(0, 1)$ to $(0, 0)$.
**Solution:**

$$\int_C x^4 dx + xy dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^1 \int_0^{1-x} (y - 0) dy dx = \int_0^1 \left[\frac{1}{2}y^2\right]_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 (1 - x)^2 dx = -\frac{1}{6}(1 - x)^3\Big|_0^1 = \boxed{\frac{1}{6}}.$$

(2) Evaluate $\int_C \left(3y - e^{\sin x}\right) dx + \left(7x + \sqrt{y^4+1}\right) dy$ where $C$ is the circle $x^2+y^2=9$.

$$\int_C \left(3y - e^{\sin x}\right) dx + \left(7x + \sqrt{y^4+1}\right) dy = \iint \times_D \left[\frac{\partial}{\partial x}\left(3y - e^{\sin x}\right) - \frac{\partial}{\partial y}\left(7x + \sqrt{y^4+1}\right)\right] dA$$
$$= \int_0^{2\pi}\int_0^3 (7-3) r dr d\theta = 2\pi \cdot 2r^2\Big|_0^3 = \boxed{36\pi}.$$

## 13.5 Flux integrals

These are also called surface integrals, and they extend the concept of a line integral. Consider the surface, $S$, defined by $z = g(x,y)$, and this surface will project onto the $xy$-plane as a rectangular domain $D$. Then

$$\iint_S f(x,y,z) dS = \iint_D f(x,y,g(x,y))\sqrt{g_x^2 + g_y^2 + 1} dA \tag{5}$$

Ex: Evaluate $\iint_S y dS$ where $S$ is $z = x + y^2$; $0 \le x \le 1$, $0 \le y \le 2$.

**Solution:**

$$\iint_S y dS = \iint_D y\sqrt{1+g_x^2+g_y^2} dA = \int_0^1\int_0^2 y\sqrt{1+1+4y^2} dy dx = \left(\int_0^1 dx\right)\left(\sqrt{2}\int_0^2 y\sqrt{1+2y^2} dy\right)$$
$$= \sqrt{2}\left(\frac{1}{4}\right)\frac{2}{3}\left(1+2y^2\right)^{3/2}\Big|_0^2 = \boxed{\frac{13}{3}\sqrt{2}}.$$

Ex: Compute the surface integral $\iint_S x^2 dS$, where S is $x^2+y^2+z^2=1$.

**Solution:** Lets differentiate implicitly first, and that should make things a lot easier,

$$\frac{\partial}{\partial x}\left[x^2+y^2+z^2=1\right] \Rightarrow 2x + 2z\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{x}{z}, \frac{\partial z}{\partial y} = -\frac{y}{z}.$$

Then

$$\iint_S x^2 dS = \iint_D x^2\sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}} dA = \iint_D \frac{x^2}{z} dA = 2\int_0^{2\pi}\int_0^1 \frac{r^2\cos^2\theta}{\sqrt{1-r^2}} r dr d\theta$$
$$= 2\left(\int_0^{2\pi}\frac{1}{2}\left[1+\cos 2\theta\right] d\theta\right)\left(\int_1^0 \frac{-1}{2}\frac{1-u}{\sqrt{u}} du\right) = \frac{2}{2}(2\pi)\left[u^{1/2} - \frac{1}{3}u^{3/2}\right]_1^0 = \boxed{\frac{4\pi}{3}}.$$

<u>Surface integral of a vector field</u>

Just like with line integrals, for surface integrals we integrate along gradients: $\iint_S F\cdot dS = \iint_D F\cdot\nabla f dA$ where $f(x,y,z) = z - g(x,y)$. Notice that

$$F\cdot\nabla f = \langle P,Q,R\rangle\cdot\langle f_x,f_y,f_z\rangle = \langle P,Q,R\rangle\cdot\langle -g_x,-g_y,1\rangle = -Pg_x - Qg_y + R = -P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R$$

therefore

$$\iint_S F\cdot dS = \iint_D F\cdot\nabla f dA = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA. \tag{6}$$

Ex: Evaluate $\iint_S F \cdot dS$ where $F(x,y,z) = y\hat{\mathbf{i}} + x\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and $S$ is the boundary of the solid between $z = 1 - x^2 - y^2$ and $z = 0$.

**Solution:**

$$\iint_S F \cdot dS = \iint \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA = \iint_D \left[-y(-2x) - x(-2y) + 1 - x^2 - y^2\right] dA$$

$$= \iint_D \left(1 + 4xy - x^2 - y^2\right) dA = \int_0^{2\pi} \int_0^1 \left(1 + 4r^2 \cos\theta \sin\theta - r^2\right) r dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left(r - r^3 + 4r^3 \cos\theta \sin\theta\right) dr d\theta = \int_0^{2\pi} \left(\frac{1}{4} + \cos\theta \sin\theta\right) d\theta = \frac{1}{4}(2\pi) + 0 = \boxed{\frac{\pi}{2}}.$$

## 13.6 Stoke's theorem

This is an extension to Green's theorem. Basically Green's theorem picks out the $z$-component of Stoke's.

**Theorem 3.** *Let $S$ be an oriented piecewise-smooth surface that is bounded by a simple, closed piecewise-smooth boundary curve $C$ with positive orientation. Let $F$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ containing $S$. Then*

$$\int_C F \cdot dr = \iint_S (\nabla \times F) \cdot dS = \iint_D (\nabla \times F) \cdot (\nabla f) dA. \tag{7}$$

Ex: Evaluate $\int_C F \cdot dr$ where $F(x,y,z) = -y^2\hat{\mathbf{i}} + x\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$ and $C$ is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$.

**Solution:** We could do this directly, but lets see how Stoke's theorem works.

First lets calculate the curl of $F$, $\nabla \times F = (1 + 2y)\hat{\mathbf{k}} = \langle 0, 0, 1 + 2y\rangle$. Then

$$\int_C F \cdot dr = \iint_D (\nabla \times F) \cdot dS = \iint_D (1 + 2y) dA = \int_0^{2\pi} \int_0^1 (1 + 2r\sin\theta) r dr d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2}r^2 + \frac{2}{3}r^3 \sin\theta\right]_0^1 d\theta = \int_0^{2\pi} \left(\frac{1}{2} + \frac{2}{3}\sin\theta\right) d\theta = \frac{1}{2}(2\pi) + 0 = \boxed{\pi}.$$

Ex: Use Stoke's theorem to compute $\iint_S (\nabla \times F) \cdot dS$ where $F(x,y,z) = xz\hat{\mathbf{i}} + yz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}$ and $S$ is part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ above $z = 0$.

**Solution:** The cylinder and sphere intersect at $z = \sqrt{3}$ in a circle, so we integrate over $C$ represented by

$$\vec{r}(t) = \langle \cos t, \sin t, \sqrt{3}\rangle \Rightarrow \vec{r}\,'(t) = \langle -\sin t, \cos t, 0\rangle$$

Then

$$\iint_S (\nabla \times F) \cdot dS = \int_C F \cdot dr = \int_0^{2\pi} F(r(t)) \cdot r'(t) dt = \int_0^{2\pi} \langle \sqrt{3}\cos t, \sqrt{3}\sin t, \cos t \sin t\rangle \cdot \langle -\sin t, \cos t, 0\rangle$$

$$= \int_0^{2\pi} (-\sqrt{3}\cos t \sin t + \sqrt{3}\sin t \cos t) dt = \boxed{0}.$$