13.7 Divergence theorem

The divergence theorem takes Stoke's theorem to surface integrals.

Theorem 1. *Let E be a simple solid region and let S be the boundary of E, with positive orientation. Let F be a vector field whose component functions have continuous partial derivatives on an open region containing E. Then*

$$\iint_S F \cdot dS = \iiint_E \nabla \cdot F dV. \tag{1}$$

Ex: Find the flux of the vector field $F(x, y, z) = z\hat{\mathbf{i}} + y\hat{\mathbf{j}} + x\hat{\mathbf{k}}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution:

$$\iint_S F \cdot dS = \iiint_B \nabla \cdot F dV = \iiint_B dV = \int_0^{\pi} \int_0^{2\pi} \int_0^1 \rho^2 \sin\phi d\rho d\theta d\phi = \boxed{\frac{4}{3}\pi}.$$

Ex: Evaluate $\iint_S F \cdot dS$ where $F(x, y, z) = xy\hat{\mathbf{i}} + \left(y^2 + e^{xz^2}\right)\hat{\mathbf{j}} + \sin(xy)\hat{\mathbf{k}}$ and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = y = 0$, $y + z = 2$.

Solution:

$$\iint_S F \cdot dS = \iiint_E \nabla \cdot F dV = \iiint_E 3y dV = 3\int_{-1}^1 \int_0^{1-x^2} \int_0^{2-z} y dy dz dx$$

$$= 3\int_{-1}^1 \int_0^{1-x^2} \frac{1}{2}(2-z)^2 dz dx = \frac{3}{2}\int_{-1}^1 \left[-\frac{1}{3}(2-z)^3\right]_0^{1-x^2} dx$$

$$= -\frac{1}{2}\int_{-1}^1 \left[(x^2+1)^3 - 8\right] dx = -\int_0^1 \left(x^6 + 3x^4 + 3x^2 - 7\right) dx = \boxed{\frac{184}{35}}.$$