SECTION 9.5 LINES IN $\mathbb{R}^3$ (CONTINUED)

Ex: (a) Find a vector equation and parametric equation for the line that passes through the point $(5, 1, 3)$ and is parallel to $\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$.

Solution: $\vec{r}_0 = \langle 5, 1, 3\rangle$ and $\vec{v} = \langle 1, 4, -2\rangle$, so $\vec{r} = (5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}) + t(\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$. Then

$$\vec{r} = (5+t)\hat{\mathbf{i}} + (1+4t)\hat{\mathbf{j}} + (3-2t)\hat{\mathbf{k}} \qquad \text{and} \qquad x = 5+t,\ y = 1+4t,\ z = 3-2t.$$

(b) Find two other points on the line.

Solution: If $t = 1$, the point is $(6, 5, 1)$ and if $t = -1$, the point is $(4, -3, 5)$.

Notice that the equation is not necessarily unique since we may pick any point and any vector in the proper direction, however it does give us a unique line. Are there other ways we can write this equation? If $a, b, c \neq 0$,

$$t = \boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}} \tag{1}$$

These are called the symmetric equations.

What happens if, for example, $a = 0$? Then $x = x_0$ and we are left with $(y - y_0)/b = (z - z_0)/c$, which means the line L lies on the vertical plane $x = x_0$.

Ex: (a) Find parametric and symmetric equations of the line that passes through $A(2, 4, -3)$ and $B(3, -2, 1)$.

Solution: Notice that $\vec{v} = \vec{AB} = \vec{B} - \vec{A} = \langle 1, -5, 4\rangle$ and let $P_0 = A$. Then

$$x = 2+t,\ y = 4-5t,\ z = -3+4t \qquad \text{and} \qquad \frac{x-2}{1} = \frac{y-4}{-5} = \frac{z+3}{4}.$$

(b) At what point does this line intersect the xy-plane?

Solution: The xy-plane is represented by $z = 0$, then $x - 2 = (4 - y)/5 = 3/4 \Rightarrow x = 11/4$ and $y = 1/4$, so the line intersects the xy-plane at point $(11/4, 1/4, 0)$.

Ex: Show that the lines L_1 and L_2 with parametric equations

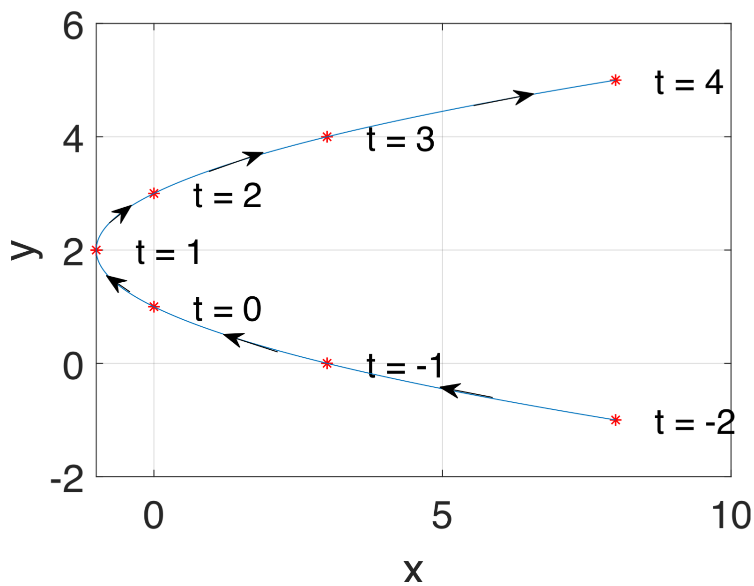
$$L_1: \quad x = 1+t,\ y = -2+3t,\ z = 4-t;$$
$$L_2: \quad x = 2s,\ y = 3+s,\ z = -3+4s.$$

are skew; i.e., they do not intersect and are not parallel (and therefore do not lie in the same plane).

Solution: The lines are not parallel because $\langle 1, 3, -1\rangle$ and $\langle 2, 1, 4\rangle$ are not parallel since their cross product will not be zero and they are not proportional to each other.

Further, if L_1 and L_2 intersected, there would be a t and s such that $1 + t = 2s$, $-2 + 3t = 3 + s$, and $4 - t = -3 + 4s$. Notice that solving the first two equations gives us $t = 11/5$ and $s = 8/5$, which doesn't satisfy the third equation.

In addition to lines we can sketch parametric curves.



Ex: Sketch and identify the curve defined by $x = t^2 - 2t$, $y = t + 1$.

Solution: One thing we can do is write out a table.

t	-2	-1	0	1	2	3	4
x	8	3	0	-1	0	3	8
y	-1	0	1	2	3	4	5

From this we can start seeing a pattern. However, there might be an easier way to identify and sketch the curve. It won't always work, but in this case it does.

Is there a way we can write this in terms of x as a function of y? We can solve for $t = t - 1 \Rightarrow x = (y-1)^2 - 2(y-1) = y^2 - 4y + 3$. So its a parabola in the x-direction.

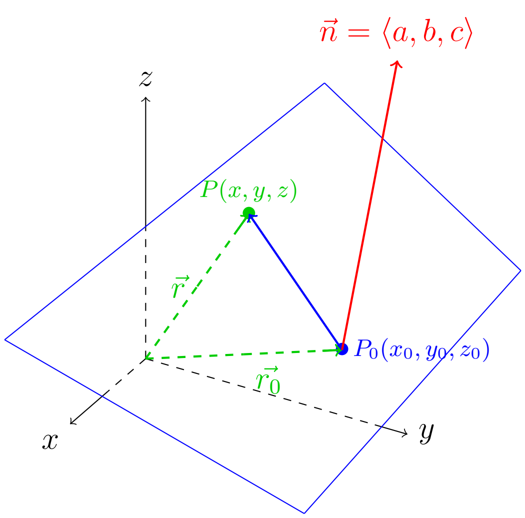
With parametric curves it is not enough to sketch the curve; it will also have a direction, with increasing t, as indicated by the arrows.

In general, the parametric equation $x = f(t)$, $y = g(t)$, $a \le t \le b$ has an initial point $(f(a), g(a))$ and a terminal point $(f(b), g(b))$.

Ex: What curve is represented by $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$? What about $x = \cos 2t$, $y = \sin 2t$.

Solution: For both of these we will get circles, however for the second one the parametric equation goes around the circle twice.

Section 9.6 Planes in $\mathbb{R}^3$



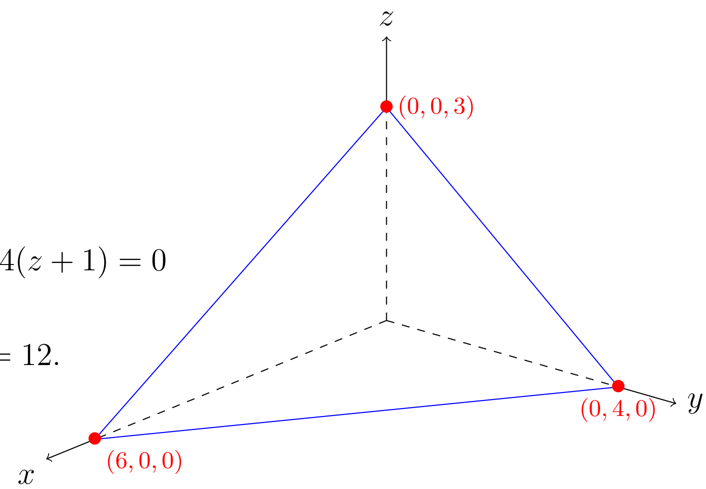
Recall that with a line we needed a point and a direction. This direction came from two points on the line. What should we do for a plane? Notice that we don't have a unique direction on the plane itself, but a vector normal to the plane does give us a unique direction (up to a constant multiple).

Notice that $\vec{P_0P} = \vec{r} - \vec{r_0}$ and that $\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$, which means $\boxed{\vec{n} \cdot \vec{r}} = \vec{n} \cdot \vec{r_0}$. This is called the vector equation of the plane, although the way we have it written is in its short form. We will see examples of writing it out. We continue simplifying, $\vec{n} \cdot (\vec{r} - \vec{r_0}) = \langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle$ to get

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \quad (2)$$

is called the scalar equation of the plane through point $P_0(x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c\rangle$.

Ex: Find an equation of the plane through the point $(2, 4, -1)$ with normal vector $\vec{n} = \langle 2, 3, 4\rangle$. Find the intercepts and sketch.



Solution: From $\vec{n}$, $a = 2$, $b = 3$, $c = 4$. So,

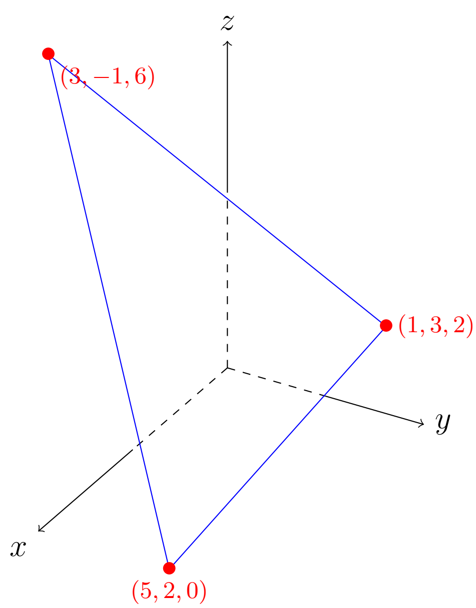
$$2(x-2) + 3(y-4) + 4(z+1) = 0$$

but this can also be re-written as

$$2x + 3y + 4z = 12.$$

From this equation finding the intercepts are easy. To find each intercept we simply set the other two variables to zero: $(6, 0, 0)$, $(0, 4, 0)$, and $(0, 0, 3)$.

The equation we wrote down for the plane is called the linear equation. In general $ax + by + cz = d$ where $d = ax_0 + by_0 + cz_0$.



Ex: Find an equation of the plane that passes through points $P(1, 3, 2)$, $Q(3, -1, 6)$, $R(5, 2, 0)$.

Solution: Notice that $\vec{PQ} = \langle 2, -4, 4\rangle$ and $\vec{PR} = \langle 4, -1, -2\rangle$. Then $\vec{n} = \vec{PQ} \times \vec{PR} = \langle 12, 20, 14\rangle$. This gives us the equation for the plane

$$12(x-1) + 20(y-3) + 14(z-2) = 0 \Rightarrow 6x + 10y + 7z = 50.$$

Ex: Find the point at which the line with parametric equation $x = 2 + 3t$, $y = -4t$, $z = 5 + t$ intersects the plane $4x + 5y - 2z = 18$.

Solution: If they intersect, there must be some point that is common. Another way to say this is, there must be some t that satisfies both equations. So, lets plug in the x, y, z from the parametric equation of the line into the equation of the plane:

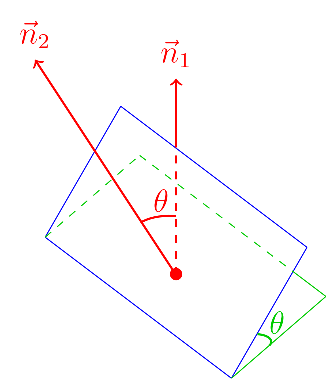
$$4(2+3t) + 5(-4t) - 2(5+t) = 18 \Rightarrow -10t = 20 \Rightarrow t = -2$$

So, the point of intersection is $(-4, 8, 3)$.

Definition 1. Two planes are parallel if their normal vectors are parallel.

For example, the planes $x + 2y - 3z = 4$ and $2x + 4y - 6z = 3$ are parallel because their normal vectors $\vec{n}_1 = \langle 1, 2, -3\rangle$ and $\vec{n}_2 = \langle 2, 4, -6\rangle$ are parallel.

Definition 2. If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors.



Ex: a) Find the angle between $x + y + z = 1$ and $x - 2y + 3z = 1$.

Solution: $\vec{n}_1 = \langle 1, 1, 1\rangle$ and $\vec{n}_2 = \langle 1, -2, 3\rangle$. Then we get

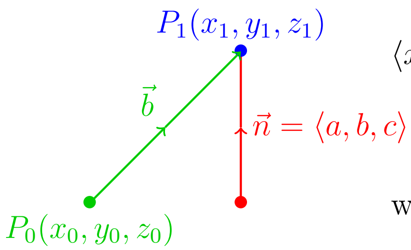
$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\|\|\vec{n}_2\|} = \frac{2}{\sqrt{42}} \Rightarrow \theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right).$$

b) Find symmetric equations for the line of intersection.

Solution: First we need to find a point on the line; i.e., some point (any point) that is common between the two planes. Lets pick $(1,0,0)$. Now we need a vector parallel to the line, which we can get by simply taking the cross product of the two normal vectors

$$\vec{n}_1 \times \vec{n}_2 = \langle 5, -2, -3 \rangle \Rightarrow \frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3}.$$

Ex: Find a formula for the distance D from $P_1(x_1, y_1, z_1)$ to $ax + by + cz - d = 0$.



Solution: If P_0 is an arbitrary point on the plane, and] $\vec{b} = \vec{P_0P_1} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$ then

$$D = |\text{comp}_{\vec{n}}\vec{b}| = \frac{|\vec{n} \cdot \vec{b}|}{\|\vec{n}\|}, \tag{3}$$

where $\vec{n} = \langle a, b, c \rangle$, so

$$D = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}.$$

Since P_0 lies on the plane, if we define the plane from this point $ax_0 + by_0 + cz_0 = d$, then this simplifies to

$$D = \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}. \tag{4}$$

This gives us an easy way to remember the distance formula. We don't have to re-derive it every time. Basically, you just plug the values of the point P_1 into the equation for a plane into the numerator, and plug in the normal vector (i.e., the coefficients of the equation of your plane) into the denominator.

Ex: Find the distance between planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.

Solution: First, note that the planes are parallel because their normal vectors $\langle 10, 2, -2 \rangle$ and $\langle 5, 1, -1 \rangle$ are parallel. Now we just choose a point on one of the plane and find the distance from that point to the other plane. Let $P_1 = (0, 1, 0)$, which is the easiest point to choose from the second plane. Now we find the distance to the first plane

$$D = \frac{|10 \cdot 0 + 2 \cdot 1 - 2 \cdot 0 - 5|}{\sqrt{10^2 + 2^2 + (-2)^2}} = \frac{3}{\sqrt{108}} = \frac{3}{6\sqrt{3}}.$$

Ex: Find the distance between skew lines:

$$L_1: \quad x_1 = 1 + t,\ y_1 = -2 + 3t,\ z_1 = 4 - t$$
$$L_2: \quad x_2 = 2s,\ y_2 = 3 + s,\ z_2 = -3 + 4s.$$

Solution: We know how to do this for planes, so why not convert the lines into planes. To do this we need to find the mutual normal

$$\langle 1, 3, -1 \rangle \times \langle 2, 1, 4 \rangle = \langle 13, -6, -5 \rangle.$$

Notice that we need only a plane from one of the lines because we could take any point from the other line and use that to calculate the distance. So, lets take the second line to determine our plane. If we let $s = 0$ we get the point $(0, 3, -3)$. Then

$$13(x - 0) - 6(y - 3) - 5(z + 3) = \boxed{13x - 6y - 5z + 3 = 0}.$$

Now pick a point on the other line: $t = 0$ gives $(1, -2, 4)$, then the distance is

$$D = \frac{13 \cdot 1 - 6 \cdot (-2) - 5 \cdot 4 + 3}{\sqrt{13^2 + (-6)^2 + (-5)^2}} = \frac{8}{\sqrt{230}}.$$

A quadratic surface in general is satisfied by

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0, \tag{5}$$

but by translation and rotation it can be simplified to

$$Ax^2 + By^2 + Cz^2 + J = 0, \qquad \text{or} \tag{6}$$

$$Ax^2 + By^2 + Iz = 0. \tag{7}$$

Ex: Sketch $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$.

Solution: If we sketch the three traces, we can get the full picture. xy-plane: $x^2 + y^2/9 = 1$, xz-plane: $x^2 + z^2/4 = 1$, and yz-plane: $y^2/9 + z^2/4 = 1$. Since all of these are ellipses, this will be an Ellipsoid.