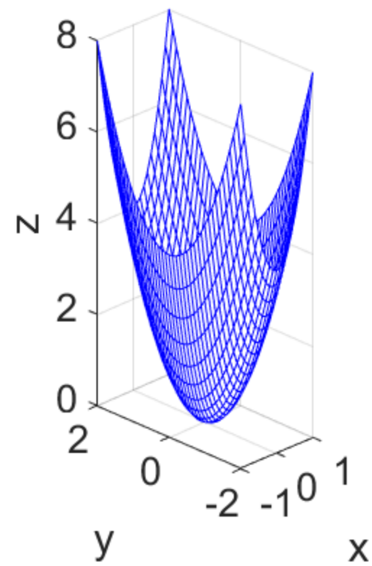
SECTION 9.7 QUADRATIC SURFACES (CONTINUED)



Ex: Sketch $z = 4x^2 + y^2$

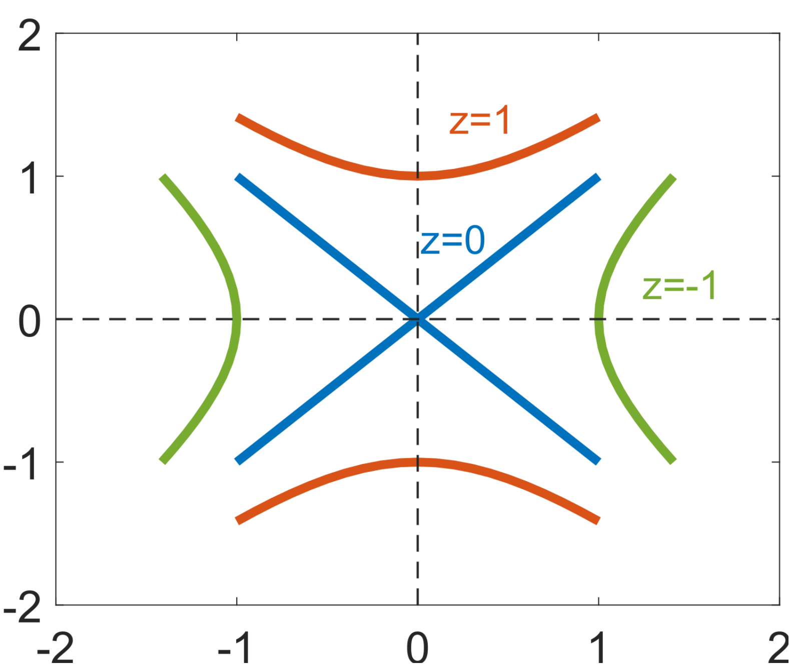
Solution: What problem do we run into in the xy-plane?

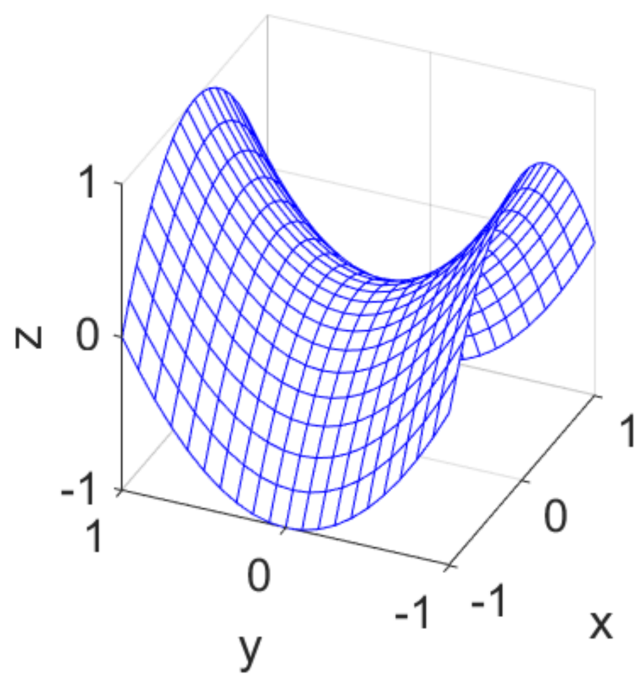
Notice that if $z = 0$, then $x = y = 0$, and $z \geq 0$, so lets take some $k \geq 0$. Now if we look at $z = k$ we get the equation $4x^2 + y^2 = k$, which is an ellipse.

Then in the xz-plane we get $y = 4x^2$, and similarly in the yz-plane we get $z = y^2$; both of which are parabolas. So, this is an elliptic paraboloid.

Ex: Sketch $z = y^2 - x^2$.

Solution: Notice that unlike the previous problem, z can be negative and $z = 0$ isn't an issue. So we have three cases, $z = 0 \Rightarrow y = \pm x$, $z = 1 \Rightarrow y^2 - x^2 = 1$, $z = -1 \Rightarrow y^2 - x^2 = -1 \Rightarrow x^2 - y^2 = 1$, which gives us hyperbolas. For this problem it is useful to sketch the trace (plot on the left). Then for





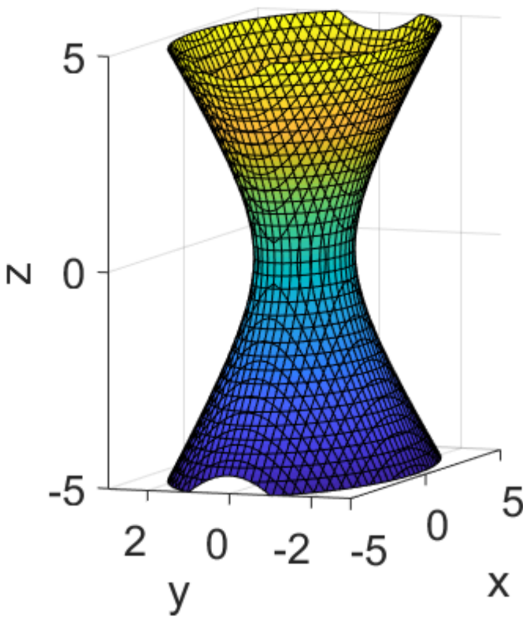
the other two directions we have $z = -x^2$ in the xz-plane and $z = y^2$ in the yz-plane, which are paraboloas. So we have a hyperbolic paraboloid also known as a saddle.

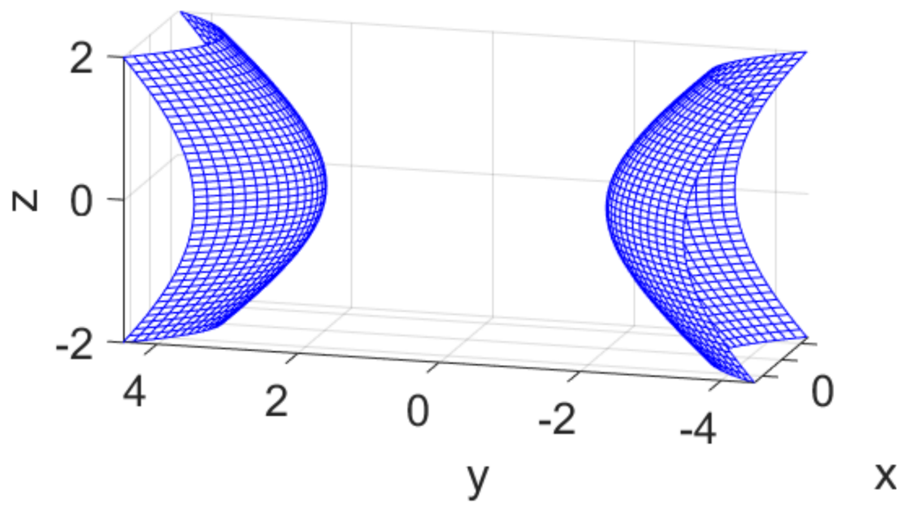
Ex: Sketch $x^2/4 + y^2 - z^2/4 = 1$.

Solution: The xy-plane gives us $x^2/4+y^2 = 1$, xz-plane: $x^2/4-z^2/4 = 1$, and yz-plane: $y^2-z^2/4 = 1$. So we have an ellipse in the xy-plane and hyperbolas in the other, so this is a Hyperboloid. Notice in the plot, that the hyperboloid is connected, and therefore of one sheet (plot on the left).

Ex: Sketch $4x^2 - y^2 + 2z^2 + 4 = 0$.

Solution: In standard form this is $-x^2 + y^2/4 - z^2 = 1$. Then for the xy-plane we get $-x^2+y^2/4 = 1$. This is a hyperbola, but notice that $|y| \geq 2$, otherwise x would be imaginary. In the xz-plane we get $x^2+z^2/2 = 1-k^2/4$ if we let $y^2 = k^2 \geq 4$, which is an ellipse. Finally, on the yz-plane we get $y^2/4-z^2/2 = 1$, which is a hyperbola. So we get a hyperboloid once again, however since it is not connected this will be of two sheets (plot on the right).





Ex: Classify the quadratic surface $x^2 + 2z^2 - 6x - y + 10 = 0$.

Solution: Notice that this is not in standard form. Everything is fine except the x portion. If we complete the square we get

$$(x-3)^2 + 2z^2 - y + 1 = 0.$$

So, this has a critical point of $(3, 1, 0)$. By looking at the traces: $z = 0$: $y = (x-3)^2$ (parabola), $y = k > 1$: $(x-3)^2 + 2z^2 = k-1$ (ellipse), and $x = 3$: $y = 2z^2 + 1$ (parabola), we see that it is an elliptic paraboloid.

Section 10.1 Vector functions

A vector valued function is a vector where each component is a function: $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$.

Ex: If $\vec{r}(t) = \langle t^3, \ln(3-t), \sqrt{t}\rangle$, then $f(t) = t^3$, $g(t) = \ln(3-t)$, and $h(t) = \sqrt{t}$. Notice that $t \in (-\infty, \infty)$ for $f(t)$, $(-\infty, 3)$ for $g(t)$, and $[0, \infty)$ for $h(t)$. So, the domain is $[0, 3)$.

Definition 1. If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to t_0} \vec{r}(t) = \left\langle \lim_{t \to t_0} f(t), \lim_{t \to t_0} g(t), \lim_{t \to t_0} h(t) \right\rangle \tag{1}$$

provided the limits of the component functions exist.

Properties of limits

(1) Sum: $lim_{t \to t_0}(\vec{r}_1(t) + \vec{r}_2(t)) = lim_{t \to t_0}\vec{r}_1(t) + lim_{t \to t_0}\vec{r}_2(t)$.
(2) Scalar multiple: $lim_{t \to t_0}(f(t)\vec{r}(t)) = (lim_{t \to t_0} f(t))\,(lim_{t \to t_0}\vec{r}(t))$.
(3) Dot product: $lim_{t \to t_0}(\vec{r}_1(t) \cdot \vec{r}_2(t)) = (lim_{t \to t_0}\vec{r}_1(t)) \cdot (lim_{t \to t_0}\vec{r}_2(t))$.
(4) Cross product: $lim_{t \to t_0}(\vec{r}_1(t) \times \vec{r}_2(t)) = (lim_{t \to t_0}\vec{r}_1(t)) \times (lim_{t \to t_0}\vec{r}_2(t))$.

Ex: Find $\lim_{t \to 0} \vec{r}(t)$ where $\vec{r}(t) = (1 + t^3)\hat{\mathbf{i}} + te^{-t}\hat{\mathbf{j}} + \sin t/t\hat{\mathbf{k}}$.

Solution:

$$\lim_{t \to 0} \vec{r}(t) = \left\langle \lim_{t \to 0}(1 + t^3), \lim_{t \to 0} te^{-t}, \lim_{t \to 0} \frac{\sin t}{t} \right\rangle$$

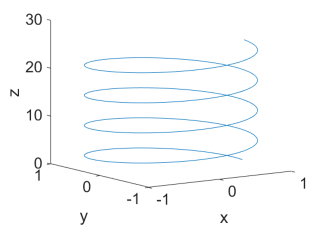
Just like with limits, we have a definition of continuity.

Definition 2. A vector function $\vec{r}$ is continuous at t_0 if $\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0)$; i.e., it is continuous if its components are continuous.

Ex: Describe the curve defined by $\langle 1 + t, 2 + 5t, -1 + 6t \rangle$.

Solution: Notice that this is just a line through $(1, 2, -1)$ with direction vector $\langle 1, 5, 6 \rangle$.

Ex: Sketch the curve for $\vec{r}(t) = \cos t\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}} + t\hat{\mathbf{k}}$.



Solution: This is a circle in the xy-plane that moves up the z direction; i.e., a helix.

Ex: Find the vector equation and parametric equation of the line through $P(1, 3, -2)$ and $Q(2, -1, 3)$.

Solution: The initial point is $P(1, 3, -2)$ and the direction vector is $\vec{v} = \langle 1, -4, 5 \rangle$. Then the vector and parametric equations are

$$\vec{r}(t) = \langle 1 + t, 3 - 4t, -2 + 5t \rangle$$
$$x = 1 + t, \qquad y = 3 - 4t, \qquad z = -2 + 5t$$

Ex: Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and plane $y + z = 2$.

Solution: Notice that $x = \cos t$ and $y = \sin t$, since the cylinder is just a circle in the xy-plane. Now we just need a parameterization of z. Since $z = 2 - y = 2 - \sin t$. Then

$$\vec{r}(t) = \cos t\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}} + (2 - \sin t)\hat{\mathbf{k}}; 0 \le t \le 2\pi.$$

Theorem 1. *If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ and f, g, h are differentiable, then*

$$\vec{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle. \tag{2}$$

What does a derivative represent? In 2-D it is the slope of the tangent, so in 3-D it is the direction vector of the tangent line.

Ex: (a) Find the derivative of $\vec{r}(t) = (1+t^3)\hat{\mathbf{i}} + te^{-t}\hat{\mathbf{j}} + \sin(2t)\hat{\mathbf{k}}$

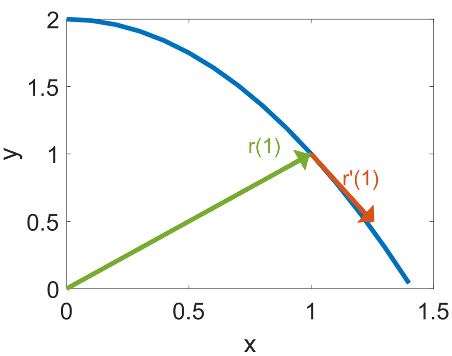
Solution:

$$\vec{r}'(t) = 3t^2\hat{\mathbf{i}} + (1-t)e^{-t}\hat{\mathbf{j}} + 2\cos(2t)\hat{\mathbf{k}}.$$

(b) Find the unit tangent vector at the point where $t = 0$.

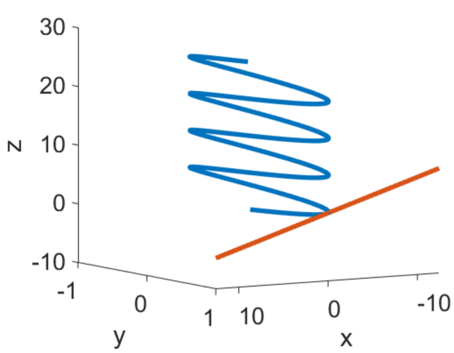
Solution: $\vec{r}(0) = \hat{\mathbf{i}}$ and $\vec{r}'(0) = \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, so the unit tangent vector at point $(1, 0, 0)$ is

$$T(0) = \frac{\vec{r}'(0)}{\|\vec{r}'(0)\|} = \frac{1}{\sqrt{5}}\hat{\mathbf{i}} + \frac{2}{\sqrt{5}}\hat{\mathbf{j}}$$



Ex: For the curve $\vec{r}(t) = \sqrt{t}\hat{\mathbf{i}} + (2-t)\hat{\mathbf{j}}$, find $\vec{r}'(t)$ and sketch the position vector $\vec{r}(1)$ and the tangent vector $\vec{r}'(1)$.

Solution: $\vec{r}'(t) = \langle 1/2\sqrt{t}, -1\rangle$, so $\vec{r}'(1) = \langle 1/2, -1\rangle$. For the sketch, notice that this is a curve on the xy-plane and $y = 2 - x^2$ with $x \geq 0$.



Ex: Find the parametric equation for the tangent line to the helix $x = 2\cos t$, $y = \sin t$, and $z = t$ at point $(0, 1, \pi/2)$.

Solution: We first notice that $t = \pi/2$, then $\vec{r}'(t) = \langle -\sin t, \cos t, 1\rangle$, and $\vec{r}'(\pi/2) = \langle -2, 0, 1\rangle$. So, the tangent line is the line through point $(0, 1, \pi/2)$ and parallel to $\langle -2, 0, 1\rangle$; i.e., the tangent line has the parametric form $x = -2t$, $y = 1$, and $z = \pi/2 + t$.

Ex: Determine whether $\vec{r}(t) = \langle 1+t^3, t^2\rangle$ is smooth ($\vec{r}'(t) \neq 0$ and continuous).

Solution: First we take the derivative $\vec{r}'(t) = \langle 3t^2, 2t\rangle$, which is zero at $t = 0$, so it is not smooth as it has a cusp at $(1, 0)$. But it is smooth at all other points, and hence is called piecewise smooth.

Ex: Show that if $\|\vec{r}(t)\| = c$ where c is a constant, then $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$ for all t.

Solution: Since $\vec{r}(t) \cdot \vec{r}(t) = \|\vec{r}(t)\|^2 = c^2$, the derivative is

$$\frac{d}{dt}\left[\vec{r}(t) \cdot \vec{r}(t)\right] = \vec{r}'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}'(t) = 2\vec{r}'(t) \cdot \vec{r}(t) = 0.$$

Therefore, $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$.

Theorem 2.

$$\int_a^b r\vec{(t)}dt = \left(\int_a^b f(t)dt\right)\hat{\boldsymbol{\imath}} + \left(\int_a^b g(t)dt\right)\hat{\boldsymbol{\jmath}} + \left(\int_a^b h(t)dt\right)\hat{\boldsymbol{k}} \tag{3}$$

Ex: If $\vec{r}(t) = (2\cos t)\hat{\mathbf{i}} + (\sin t)\hat{\mathbf{j}} + (2t)\hat{\mathbf{k}}$, then

$$\int \vec{r}(t)dt = (2\sin t)\hat{\mathbf{i}} + (-\cos t)\hat{\mathbf{j}} + (t^2)\hat{\mathbf{k}} + C,$$

and

$$\int_0^{\pi/2} \vec{r}(t)dt = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \frac{\pi^2}{4}\hat{\mathbf{k}}.$$

Recall that all of these concepts are used for particles of motion.

- $\vec{r}(t)$ is the position vector of a moving object,
- $\vec{v} = d\vec{r}/dt$ is the velocity,
- $\|\vec{v}\|$ is the speed,
- $\vec{v}/\|\vec{v}\|$ gives us the direction, and
- $\vec{a} = d\vec{v}/dt = d^2\vec{r}/dt^2$ is the acceleration.

10.4 Curvature

Lets first talk about arc length. What's the formula for the length of a straight line?

$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}$ for $x_0 \leq x \leq x_1$. We can approximate the length of a curve by using our straight line formula. To get a better approximation we just use more and more points. This gives us a sum

$$\sum_{n=1}^{N} \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$$

Then taking the limit of this sum gives us the integral $\int_a^b \sqrt{dx^2 + dy^2 + dz^2}$, however this integral isn't in the form that we are used to. We can't integrate this as is. Lets multiply the integrand by dt/dt to get

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2} dt = \int_a^b \|\vec{r}'(t)\| dt. \qquad (4)$$

Ex: Find the length of the arc $\vec{r}(t) = (\cos t)\hat{\mathbf{i}} + (\sin t)\hat{\mathbf{j}} + t\hat{\mathbf{k}}$ from point $(1, 0, 0)$ to $(1, 0, 2\pi)$.

$$\|\vec{r}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}$$

$$\Rightarrow L = \int_0^{2\pi} \|\vec{r}'(t)\| dt = \int_0^{2\pi} \sqrt{2} dt = 2\sqrt{2}\pi.$$