10.4 Curvature (continued)

We can also write the arclength as a function by using a dummy variable:

$$s(t) = \int_a^t \|\vec{r}\,'(u)\| du; \qquad \text{and} \qquad \frac{ds}{dt} = \|\vec{r}\,'(t)\| \tag{1}$$

Ex: Reparametrize $\vec{r}(t) = (\cos t)\hat{\mathbf{i}} + (\sin t)\hat{\mathbf{j}} + t\hat{\mathbf{k}}$ with respect to the arclength measured from $(1,0,0)$ in the direction of increasing t.

Solution: The initial point $(1,0,0)$ corresponds to $t=0$, so

$$s(t) = \int_0^t \|\vec{r}\,'(u)\| du = \int_0^t \sqrt{2} du = \sqrt{2}t \Rightarrow t = s/\sqrt{2},$$

therefore reparametrizing gives us

$$\vec{r}(s) = (\cos(s/\sqrt{2}))\hat{\mathbf{i}} + (\sin(s/\sqrt{2}))\hat{\mathbf{j}} + \frac{s}{\sqrt{2}}\hat{\mathbf{k}}$$

Recall that the tangent vector is given by $\vec{r}\,'(t)$, then the unit tangent vector is

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}. \tag{2}$$

From this, we can find the curvature, which is how quickly the curve changes direction. So, given a section of the curve, we want to know the change in direction (i.e., change in T). In class we sketched this and showed that we can approximate the curvature as $\Delta T/\Delta s$, and taking the limit gives us $\kappa = \|dT/ds\|$. However, a better formula comes from the simplification

$$\kappa = \left\|\frac{dT}{ds}\right\| = \left\|\frac{dT/dt}{ds/dt}\right\| = \boxed{\frac{\|T'(t)\|}{\|\vec{r}\,'(t)\|}}. \tag{3}$$

Ex: Show that the curvature of a circle of radius a is $1/a$.

Solution: A circle of radius a can be written as $\vec{r} = (a\cos t)\hat{\mathbf{i}} + (a\sin t)\hat{\mathbf{j}}$. Then the derivative gives us

$$\vec{r}\,'(t) = (-a\sin t)\hat{\mathbf{i}} + (a\cos t)\hat{\mathbf{j}} \Rightarrow \|\vec{r}\,'(t)\| = a,$$

so

$$T = (-\sin t)\hat{\mathbf{i}} + (\cos t)\hat{\mathbf{j}} \Rightarrow T' = (-\cos t)\hat{\mathbf{i}} + (-\sin t)\hat{\mathbf{j}} \Rightarrow \|T'(t)\| = 1,$$

then

$$\kappa = \frac{\|T'(t)\|}{\|\vec{r}\,'(t)\|} = \frac{1}{a}. \tag{4}$$

Another formula for curvature is

$$\kappa(t) = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}. \tag{5}$$

We can derive this by writing $T = \vec{r}\,'/\|\vec{r}\,'\|$ and $\|\vec{r}\,'\| = ds/dt$, where s is the arc length. Then

$$\vec{r}\,' = \|\vec{r}\,'\|T = \frac{ds}{dt}T \Rightarrow \vec{r}\,'' = \frac{d^2s}{dt^2}T + \frac{ds}{dt}T'.$$

Since $T \times T = 0$,

$$\vec{r}\,' \times \vec{r}\,'' = \left(\frac{ds}{dt}\right)^2 (T \times T').$$

By definition $\|T\| = 1$ for all t (i.e., a constant) so T and T' are orthogonal, and hence $\|T \times T'\| = \|T\|\|T'\| = \|T'\|$. So,

$$\|\vec{r}\,' \times \vec{v}\,''\| = \left(\frac{ds}{dt}\right)^2 \|T'\| \Rightarrow \|T'\| = \frac{\|\vec{r}\,' \times \vec{r}\,''\|}{(ds/dt)^2} \Rightarrow \kappa = \frac{\|T'\|}{\|\vec{r}\,'\|} = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$$

Ex: Find the curvature of the twisted cubic $\vec{r} = \langle t, t^2, t^3\rangle$ at $(0,0,0)$.

Solution: Lets calculate all the elements we need first

$$\vec{r}\,' = \langle 1, 2t, 3t^2\rangle \Rightarrow \vec{r}\,'' = \langle 0, 2, 6t\rangle \Rightarrow \|\vec{r}\,'(t)\| = \sqrt{1 + 4t^2 + 9t^4},$$

and

$$\vec{r}\,' \times \vec{r}\,'' = \langle 6t^2, -6t, 2\rangle \Rightarrow \|\vec{r}\,' \times \vec{r}\,''\| = 2\sqrt{9t^4 + 9t^2 + 1}$$

then

$$\kappa(t) = \frac{\|\vec{r}\,' \times \vec{r}\,''\|}{\|\vec{r}\,'\|^3} = \frac{2\sqrt{9t^4 + 9t^2 + 1}}{(1 + 4t^2 + 9t^4)^{3/2}} \Rightarrow \kappa(0) = 2.$$

Ex: Find the curvature of the parabola $y = x^2$ at points $(0,0)$, $(1,1)$, and $(2,4)$.

Solution: Notice that we can choose any parametrization for x, so lets pick the easiest, $x = t$; i.e., $\vec{r} = \langle t, t^2, 0\rangle$. We calculate all the elements

$$\vec{r}\,' = \langle 1, 2t, 0\rangle \Rightarrow \vec{r}\,'' = \langle 0, 2, 0\rangle \Rightarrow \|\vec{r}\,' \times \vec{r}\,''\| = 2$$

and

$$\|\vec{r}\,'\| = \sqrt{1 + 4t^2} \Rightarrow \kappa(t) = \frac{2}{\sqrt{1 + 4t^2}^3}$$

Then the curvature at the different points are

$$\kappa(0) = 2, \qquad \kappa(1) = \frac{2}{5^{3/2}}, \qquad \kappa(2) = \frac{2}{17^{3/2}}.$$

We can generalize the previous example as follows,

$$\vec{r} = \langle x, x^2, 0\rangle = \vec{r}\,' \times \vec{r}\,'' = f''(x)\hat{\mathbf{k}}$$

and $\|\vec{r}\,'\| = \sqrt{1 + f'(x)^2}$, then the curvature for a 2-D curve simplifies to

$$\kappa = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}}. \tag{6}$$

Recall that T' is perpendicular to T since $\|T\| = 1$; i.e., a constant. Further, T' points to the center of the curve, which means it is parallel to the normal vector of the plane that is tangent to the curve. When we normalize it, we call it the unit normal vector,

$$N(t) = \frac{T'(t)}{\|T'(t)\|}. \tag{7}$$

And since we are in 3-D we also have a binormal vector that is perpendicular to both N and T, which can be found by taking the cross product $B = N \times T$.

Ex: Find the unit normal vector for $\vec{r} = (\cos t)\hat{\mathbf{i}} + (\sin t)\hat{\mathbf{j}} + t\hat{\mathbf{k}}$.

Solution: As before we compute the elements we need

$$\vec{r}\,' = (-\sin t)\hat{\mathbf{i}} + (\cos t)\hat{\mathbf{j}} + \hat{\mathbf{k}} \Rightarrow \|\vec{r}\,'\| = \sqrt{2} \Rightarrow T(t) = \frac{\vec{r}\,'}{\|\vec{r}\,'\|} = \frac{1}{\sqrt{2}}\left[(-\sin t)\hat{\mathbf{i}} + (\cos t)\hat{\mathbf{j}} + \hat{\mathbf{k}}\right]$$

$$\Rightarrow T'(t) = \frac{1}{\sqrt{2}}\left[(-\cos t)\hat{\mathbf{i}} + (-\sin t)\hat{\mathbf{j}}\right] \Rightarrow \|T'(t)\| = \frac{1}{\sqrt{2}} \Rightarrow N(t) = \frac{T'(t)}{\|T'(t)\|} = \langle -\cos t, \sin t, 0\rangle$$

Ex: Find and sketch the osculating circle of $y = x^2$ at the origin.

Solution: This is the circle with the same curvature of the curve that just touches the curve at the point, in this case the origin, and has a center in the direction of the normal of the curve. As we computed already the curvature at the origin is $\kappa(0) = 2$, then the circle with the same curvature will have a radius of $r = 1/2$. Also, notice that at the origin we can see that the normal vector is in the positive y-direction without doing any work, since the tangent vector is along the x-axis. So, the center is at point $(0, 1/2)$, then the equation for the circle is

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4} \Rightarrow \vec{r} = \left\langle \frac{1}{2}\cos t, \frac{1}{2} + \frac{1}{2}\sin t \right\rangle$$

11.1 Functions of several variables

In this chapter we will look at functions of the type $z = f(x, y)$, which works just like functions of single variables, but now we have an ordered pair being plugged into the function.

Ex: Find the domain of the following functions and evaluate $f(3, 2)$.

(a) $f(x, y) = \sqrt{x + y + 1}/(x - 1)$

Solution: $D = \{(x, y) \in \mathbb{R}^2 | x \neq 1, x + y + 1 \geq 0\}$ and $z = f(3, 2) = \sqrt{6}/2$.

(b) $f(x, y) = x\ln(y^2 - x)$

Solution: $D = \{(x, y) \in \mathbb{R}^2 | x < y^2\}$ and $f(3, 2) = 3\ln(1) = 0$.

Ex: Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Solution:

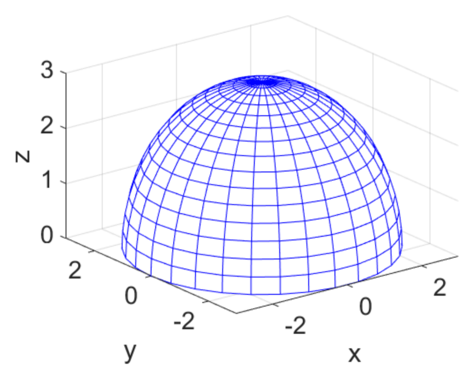
$$D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 9\} \qquad \text{and} \qquad R = \{z \in \mathbb{R}^2 | 0 \leq z \leq 3\}.$$

Ex: Sketch the graph of the function $f(x, y) = 6 - 3x - 2y$

Solution: Notice that this is just the plane $3x + 2y + z = 6$, which we sketched in Week 2.

Ex: Sketch the graph $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Solution: This gives us $x^2 + y^2 + z^2 = 9$; $z \geq 0$, which is the hemisphere on the right



Ex: Find the domain and range and sketch $h(x, y) = 4x^2 + y^2$.

Solution: This is $z = 4x^2 + y^2$, which is an elliptic paraboloid, which we sketched in Week 3.