11.1 Functions of several variables (continued)

One way we can visualize 3-D structures is through level curves.

Definition 1. Level curves of a function $f(x,y)$ are curves of equation $f(x,y)=k$, where k is a constant (in the range of f).

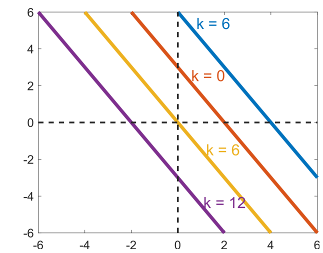
For example, ground elevation on Earth's surface is a function $z=f(x,y)$, but to illustrate this on a map contour lines are used, where curves of the same elevation are sketched.

An example of this that we did in Sec. 9.7 are the traces to figure out what the quadratic surface is.

We did an example of this in class, which I won't redraw here, but think of topographic maps.

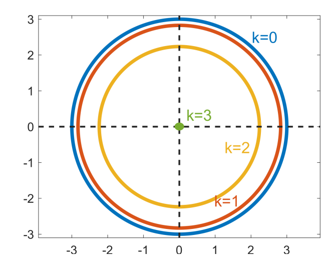
Ex: Sketch the level curves of the function $f(x,y)=6-3x-2y$ for $k=-6,0,6,12$.

Solution: The level curves are $6-3x-2y=k \Rightarrow y=-\frac{3}{2}x+(6-k)/2$, which are lines.



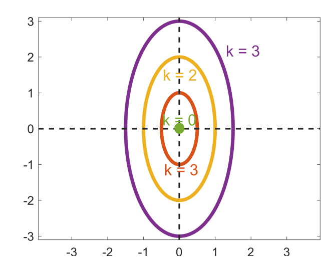
Ex: Sketch the level curves of the function $g(x,y)=\sqrt{9-x^2-y^2}$ for $k=0,1,2,3$.

Solution: We do the same thing as above: $\sqrt{9-x^2-y^2}=k \Rightarrow x^2+y^2=9-k^2$, which are circles.



Ex: Sketch some level curves of $h(x,y)=4x^2+y^2$.

Solution: We do the same thing as before: $4x^2+y^2=k \Rightarrow x^2+y^2/4=k/4$, which are ellipses.



If we have a function of three variables we can have level surfaces.

Ex: Find the domain of $f(x,y,z)=\ln(z-y)+xy\sin z$.

Solution: $D=\{(x,y,z)\in\mathbb{R}^3 : z>y\}$.

Ex: Identify the level surfaces of $f(x,y,z)=x^2+y^2+z^2$.

Solution: These will be concentric spheres centered at zero with a radius of $\sqrt{k}$.

With functions of multiple variables, we can't just take our usual limit. Our surface is made up of a bunch of different paths, so we need the limit to exist on all of those paths. There are times when it is obvious that a limit exists (continuous functions) and other times when a limit definitely doesn't exists (the singularity in the denominator overpowers the numerator). Let us first see what happens when it is unclear whether a limit exists or not (i.e., indeterminate form).

Definition 2. If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist (DNE).

Ex: Show that the following limit doesn't exist

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}.$$

Solution: First lets approach $(0, 0)$ along the x-axis; i.e., $y = 0$, then $f(x, 0) = 1 \Rightarrow f(x, y) \to 1$ as $(x, y) \to (0, 0)$ along the x-axis.
Now, approach along the y-axis: $f(0, y) = -1 \Rightarrow f(x, y) \to -1$ as $(x, y) \to (0, 0)$ along the y-axis.
And therefore, the limit DNE.

Ex: If $f(x, y) = xy/(x^2 + y^2)$, does $\lim_{(x,y)\to(0,0)} f(x, y)$ exist?
Solution: Again, lets take our two approaches.
x-axis: $\lim_{(x,y)\to(0,0)} f(x, y) = 0$.
y-axis: $\lim_{(x,y)\to(0,0)} f(x, y) = 0$.
However, we can pick one of the infinitely many directions, say along $y = x$, which gives us $\lim_{(x,y)} \to (0, 0)f(x, y) = 1/2$.
So, the limit DNE.

Ex: If $f(x, y) = xy^2/(x^2 + y^4)$, does $\lim_{(x,y)\to(0,0)} f(x, y)$ exist?
Solution: Now lets save some time and go along a generic line $y = mx$,

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{m^2x}{1 + m^4x^2} = 0.$$

However, if we take $x = y^2$, $f(y^2, y) = 1/2$, then $\lim_{(x,y)\to(0,0)} f(x, y) = 1/2$.
So, once again, the limit does not exist.

Types of problems where a limit definitely exists:

(1) I tell you *a priori* that it exists,
(2) It's a rational function with no singularities,
(3) It's a rational function with removable discontinuities, or
(4) A function that is clearly continuous everywhere.

Definition 3. A function of two variables is called continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b). \tag{1}$$

We say f is continuous on domain D if f is continuous at every point (a, b) in the domain D.

Examples of functions that are continuous everywhere are: polynomials, exponentials, sine and cosine, $tan^{-1}x$.

Ex: Evaluate $\lim_{(x,y)\to(1,2)} x^2y^3 - x^3y^2 + 3x + 2y$.

Solution: Since f is a polynomial, it is continuous everywhere, and hence we simply plug the numbers in; i.e., $\lim_{(x,y)\to(1,2)} f(x,y) = 11$.

Ex: Where is $f(x,y) = (x^2 - y^2)/(x^2 + y^2)$ is continuous?

Solution: Everywhere except $(0,0)$.

Ex: Notice that if we add in the point 0 if $(x,y) = (0,0)$,

$$g(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0); \end{cases}$$

it is still not continuous because as we showed in a previous example $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE.

Ex: On the other hand if the limit for our function does exists, then we can simply fill in that point at the limit. For example

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0,$$

then we can fill in the point at $(0,0)$ and make it continuous,

$$f(x,y) = \begin{cases} \frac{3x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0); \end{cases}$$

is continuous.

Ex: The function $h(x,y) = \tan^{-1}(y/x)$ is continuous everywhere except line $x = 0$, since y/x is a rational function and $\tan^{-1}$ is continuous everywhere.

11.3 Partial derivatives

When we take partial derivatives with respect to one variable we simply keep the other constant.

Ex: If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2,1)$ and $f_y(2,1)$.

Solution:

$$f_x = 3x^2 + 2xy^3 \Rightarrow f_x(2,1) = 16,$$
$$f_y = 3x^2y^2 - 4y \Rightarrow f_y(2,1) = 8.$$

Notation: (If $z = f(x,y)$),

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = D_x f = \partial_x f$$
$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = D_y f = \partial_y f$$