11.3 Partial derivatives (continued)

Ex: If $f(x,y) = 4 - x^2 - 2y^2$, find $f_x(1,1)$ and $f_y(1,1)$ and interpret the slopes.

Solution: Lets first find the slopes; i.e., take the derivative. $f_x = -2x \Rightarrow f_x(1,1) = -2$ and $f_y = -4y \Rightarrow f_y(1,1) = -4$.

Lets first think about what sort of shape this is. It is clearly a quadratic surface, but specifically and elliptic paraboloid. Then what do the derivatives tell us? They are the slopes of the zx and zy traces.

Ex: If $f(x,y) = \sin(x/(1+y))$, calculate $\partial f/\partial x$ and $\partial f/\partial y$.

Solution:

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right)\frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right)\frac{1}{1+y}$$
$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right)\frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = -\cos\left(\frac{x}{1+y}\right)\frac{x}{(1+y)^2}.$$

Ex: Find $\partial z/\partial x$ and $\partial z/\partial y$ if z is defined implicitly as a function of x and y in the equation $x^3 + y^3 + z^3 + 6xyz = 1$.

Solution:

$$\frac{\partial}{\partial x}\left[x^3 + y^3 + z^3 + 6xyz = 1\right] \Rightarrow 3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{-3x^2 - 6yz}{3z^2 + 6xy};$$

Similarly,

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}.$$

Ex: Find f_x, f_y, f_z if $f(x,y,z) = e^{xy}\ln z$.

Solution: $f_x = ye^{xy}\ln z,\ f_y = xe^{xy}\ln z,\ f_z = e^{xy}/z.$

We can also have higher order derivatives just like with single variable functions: $\partial_x(f_x) = f_{xx}$, $\partial_y(f_y) = f_{xy}$, $\partial_x(f_y) = f_{yx}$, and $\partial(f_y) = f_{yy}$.

Ex: Find the second partial derivatives of $f(x,y) = x^3 + x^2y^3 - 2y^2$.

Solution: Recall $f_x = 3x^2 + 2xy^3$ and $f_y = 3x^2y^2 - 4y$, then $f_{xx} = 6x + 2y^3$, $f_{xy} = 6xy^2$, $f_{yx} = 6xy^2$, $f_{yy} = 6x^2y - 4$.

We notice that the middle partials are equivalent. However, this is because the function is "nice", but it won't work for all functions.

Theorem 1. *Clairaut Suppose f is defined on a disk D that contains point (a,b). If f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a,b) = f_{yx}(a,b)$.*

This also extends to higher derivatives, for example $f_{xyy} = f_{yxy} = f_{yyx}$ if those functions are continuous.

Ex: Calculate f_{xxyz} if $f(x,y,z) = \sin(3x + yz)$.

Solution:

$$f_x = 3\cos(3x+yz) \Rightarrow f_{xx} = -9\sin(3x+yz) \Rightarrow f_{xxy} = -9z\cos(3x+yz) \Rightarrow f_{xxyz} = -9\cos(3x+yz)+9yz\sin(3x+yz).$$

Ex: Show that $u(x,y) = e^x \sin y$ is a solution to Laplace's equation $u_{xx} + u_{yy} = 0$.

Solution: All we have to do is plug into the differential equation. Notice that $u_{xx} = e^x \sin y$ and $u_{yy} = -e^x \sin y$, so it is obvious that this function satisfies the differential equation.

Ex: Verify that $u(x,y) = \sin(x - ct)$ satisfies the wave equation $u_{tt} = c^2 u_{xx}$.

Solution: We do the same exact thing as before. $u_{tt} = -c^2 \sin(x - ct)$ and $u_{xx} = -\sin(x - ct)$. And once again it is clear that this satisfies the differential equation.

11.4 Tangent planes, approximations, differentiability

Consider a surface $x = f(x,y)$, where f has continuous first partial derivatives. Just like in the previous section, we can find the tangent line in the x-direction and y-direction at point (x_0, y_0, z_0) by doing $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$. For the tangent line in the x-direction we just use rise over run to get

$$f_x(x_0, y_0) = \frac{z - z_0}{x - x_0} \Rightarrow z - z_0 = f_x(x_0, y_0)(x - x_0),$$

and similarly for the y-direction we get $z - z_0 = f_y(x_0, y_0)(y - y_0)$. Now let write down the equation of a plane,

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \Rightarrow a(x - x_0) + b(y - y_0) = z - z_0$$

where $a = -A/C$ and $b = -B/C$. Now, if we set $y = y_0$ we get $z - z_0 = a(x - x_0)$, and notice that at the tangent line in the x-direction the slope is $a = f_x(x_0, y_0)$, and similarly $b = f_y(x_0, y_0)$, then an equation of the tangent plane to the surface $z = f(x,y)$ at point (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \tag{1}$$

Ex: Find the tangent plane of the elliptic paraboloid $z = 2x^2 + y^2$ at point $(1, 1, 3)$.

Solution: Let $f(x,y) = 2x^2 + y^2$, then $f_x = 4x \Rightarrow f_x(1,1) = 4$, and $f_y = 2y \Rightarrow f_y(1,1) = 2$, and hence

$$z - 3 = 4(x - 1) + 2(y - 1).$$

What happens to the distance between the plane and the surface as we move closer to the reference point? What can we use this concept for? This works just like tangent lines in 1-D. We can approximate our quadratic surface with a tangent plane, and the approximation gets better as we move closer to the reference point.

Linear approximations

In our previous example we get our linearization of f at $(1,1)$ by solving for z:

$$L(x,y) = 4x + 2y - 3. \tag{2}$$

and we can use this for our linear approximation of f at $(1,1)$:

$$f(x,y) \approx 4x + 2y - 3. \tag{3}$$

For example, $f(1.1, 0.95) \approx 4\cdot(1.1) + 2\cdot(0.95) - 3 = 3.3$, which is close to the real value: $f(1.1, 0.95) = 3.3225$. However, $L(2,3) = 11$, but $f(2,3) = 17$, so $(2,3)$ is too far from $(1,1)$ for the linearization L to be a good approximation for f.

In general if $z_0 = f(a,b)$; i.e., $a = x_0$ and $b = y_0$, then

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \qquad \text{is the tangent plane,} \tag{4a}$$
$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \qquad \text{is the linearization,} \tag{4b}$$
$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \qquad \text{is the linear approximation,} \tag{4c}$$

We may not always have such approximations. Consider

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0); \end{cases}$$

Here $f_x(0,0) = 0$ and $f_y(0,0) = 0$, but $f(x,y) \approx 0$ and $f(x,y) = 1/2$ along the line $y = x$, so we cannot use this approximation. This is because f_x and f_y are not continuous.

Theorem 2. *If the partial derivatives f_x and f_y exist near (a,b) and are continuous at (a,b), then f is differentiable at (a,b).*

Notice that this may not be the only time f is differentiable, but continuity of f_x and f_y guarantees differentiability.

Ex: Show that $f(x,y) = xe^{x,y}$ is differentiable at $(1,0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

Solution: $f_x = e^{xy} + xye^{xy}$ and $f_y = x^2e^{xy}$ are continuous since they are products of polynomials and exponentials, so the function is differentiable. Therefore, we can calculate the linearization at $(1,0)$

$$L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) = 1 + (x-1) + y = x + y,$$

and we can use this to approximate the function near that point; i.e., $xe^{xy} = x + y$ at $(1,0)$, so

$$f(1.1, -0.1) \approx 1.1 - 0.1 = 1.$$

Just as with functions of one variable, we have differentials:

$$\frac{dz}{dx} = f_x(x,y) \Rightarrow dz = f_x(x,y)dx,$$

in the x-direction and similarly $dz = f_y(x,y)dy$ in the y-direction, so

$$dz = f_x(x,y)dx + f_y(x,y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \tag{5}$$

Ex: (a) If $z = f(x,y) = x^2 + 3xy - y^2$, find the differential dz.
Solution:

$$dz = f_x dx + f_y dy = (2x+3y)dx + (3x-2y)dy.$$

(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare Δz with dz.
Solution: $dx = 0.05$ and $dy = -0.04$, then

$$dz = (2 \cdot 2 + 3 \cdot 3) \cdot 0.05 + (3 \cdot 2 - 2 \cdot 3) \cdot (-0.04) = 0.65.$$

and

$$\Delta z = f(2.05, 2.96) - f(2,3) = 0.6449.$$

So, dz is a good approximation of Δz, and easier to compute.

Ex: The base radius and height of a right circular cone are measured as 10 cm and 25 cm respectively, with a possible error in measurement of as much as 0.1 $\mathbf{0.1}cm$ in each. Use differentials to estimate the maximum error in the calculated volume.
Solution: The volume of a right circular cone $V = \pi r^2 h/3$, then

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh = \frac{2\pi rh}{3}dr + \frac{\pi r^2}{3}dh,$$

then

$$dV = \frac{2\pi}{3}(10)(25)(0.1) + \frac{\pi}{3}(10)^2(0.1) = 20\pi \quad \text{cm}^3.$$

Recall for a single variable function, if $f(x)$ and $x = g(t)$, then $dy/dt = (dy/dx)(dx/dt)$, then for a two variable function $z = f(x, y)$ and $x = g(t)$, $y = h(t)$, this becomes

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \tag{6}$$

Ex: If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt at $t = 0$.

Solution: Let us go step by step with the derivatives $\partial z/\partial x = 2xy + 3y^4$, $\partial z/\partial y = x^2 + 12xy^3$, $dx/dt = 2\cos 2t$, and $dy/dt = -\sin t$.

Now we want to evaluate at $t = 0$, but notice that it is not absolutely necessary to substitute the functions for x and y since we can simply plug in $t = 0$, so

$$\left.\frac{dz}{dt}\right|_{t=0} = (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)\Big|_{t=0,\,x=0,\,y=1} = 6.$$

What if x and y are functions of two variables? Let $z = f(x, y)$ and $x = g(s, t)$, $y = h(s, t)$, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \tag{7a}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \tag{7b}$$

Ex: If $z = e^x \sin y$ where $x = st^2$ and $y = s^2t$, find $\partial z/\partial s$ and $\partial z/\partial t$.

Solution:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (e^x \sin y)t^2 + (e^x \cos y)(2st) = t^2e^{st^2}\sin(s^2t) + 2ste^{st^2}\cos(s^2t)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)s^2 = 2ste^{st^2}\sin(s^2t) + s^2e^{st^2}\cos(s^2t).$$

This can be extended to as many variables we want, but it won't help to memorize these formulas. Think about the concepts and just differentiate.