11.5 Chain rules (continued)

Ex: If $u = x^4 y + y^2 z^3$ where $x = rse^t$, $y = rs^2 e^{-t}$, and $z = sr^2 \sin t$. Find the value of $\partial u/\partial s$ when $r = 2$, $s = 1$, and $t = 0$.

Solution:

$$\frac{\partial u}{\partial s} = (4x^3 y)(re^t) + (x^4)(2rse^{-t}) + (2yz^3)(2rse^{-t}) + (3y^2 z^2)(r^2 \sin t).$$

When $r = 2$, $s = 1$, $t = 0$, we have $x = 2$, $y = 2$, $z = 0$, then

$$\left.\frac{\partial u}{\partial s}\right|_{r=2,s=1,t=0} = 192.$$

Ex: If $z = f(x, y)$ has continuous second order derivatives and $x = r^2 + s^2$ and $y = 2rs$, find

(a) $\partial z/\partial r$

Solution:

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = 2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y}.$$

(b) $\partial^2 z/\partial r^2$

Solution:

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r}\left[2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y}\right] = 2\frac{\partial z}{\partial x} + 2r\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + 2s\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)\\ &= 2\frac{\partial z}{\partial x} + 2r\left[(2r)\frac{\partial^2 z}{\partial x^2} + (2s)\frac{\partial^2 z}{\partial x \partial y}\right] + 2s\left[(2r)\frac{\partial^2 z}{\partial y \partial x} + (2s)\frac{\partial^2 z}{\partial y^2}\right]\\ &= 2\frac{\partial z}{\partial x} + 4r^2\frac{\partial^2 z}{\partial x^2} + 8rs\frac{\partial^2 z}{\partial x \partial y} + 4s^2\frac{\partial^2 z}{\partial y^2}.\end{aligned}$$

Implicit function theorem. We have seen how implicit differentiation works, but there is a faster formula. Suppose $y = f(x)$ then $y - f(x) = 0$. Define $F(x, y) = y - f(x) = 0$. If F is differentiable and $\partial F/\partial y \neq 0$, then

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x}\cancelto{1}{\frac{dx}{dx}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} = 0 \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Ex: Find dy/dx if $x^3 + y^3 = 6xy$.

Solution: Let $F(x, y) = x^3 + y^3 - 6xy = 0$, then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x}.$$

Ex: Find $\partial z/\partial x$ and $\partial z/\partial y$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

Solution: Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1 = 0$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy}$$

We can take derivatives in the x and y directions. What is the natural question we want to ask? How can we look at the tangent line in any direction?

Suppose we want the slope in the $\vec{u} = \langle a, b\rangle$ direction where $\|\vec{u}\| = 1$. Then if $\vec{u} = \langle 1, 0\rangle$, the slope is completely in the x-direction; i.e., f_x, otherwise if $\vec{u} = \langle 0, 1\rangle$, it is completely in the y-direction; i.e., f_y. If it's something in between, say $\langle 1/\sqrt{2}, 1/\sqrt{2}\rangle$, then the slope, using trigonometry, is $f_x/\sqrt{2} + f_y/\sqrt{2}$. So, we can write this for any general direction.

Theorem 1. *If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\vec{u} = \langle a, b\rangle$ and $D_u f(x, y) = af_x(x, y) + bf_y(x, y)$.*

Ex: Find the directional derivative $D_u f(x, y)$ if $f(x, y) = x^3 - 3xy + 4y^2$ and in the direction of $\theta = \pi/6$ to the x-axis. What is $D_u f(1, 2)$?

Solution: The first thing we want to do is figure out what the unit direction vector is. Since we are given an angle with respect to the x-axis we use $\vec{u} = \langle \cos\pi/6, \sin\pi/6\rangle = \langle\sqrt{3}/2, 1/2\rangle$. Then

$$D_u f(x, y) = f_x(x, y)\frac{\sqrt{3}}{2} + f_y(x, y)\frac{1}{2} = (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\frac{1}{2} \Rightarrow D_u f(1, 2) = \frac{13 - 3\sqrt{3}}{2}.$$

Now what does $af_x(x, y) + bf_y(x, y)$ remind us of? We can write this as $D_u f(x, y) = af_x(x, y) + bf_y(x, y) = \langle f_x(x, y), f_y(x, y)\rangle \cdot \langle a, b\rangle$.

Definition 1. If f is a function of two variables x and y, then the gradient of f is the vector function defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \frac{\partial f}{\partial x}\hat{\mathbf{\imath}} + \frac{\partial f}{\partial y}\hat{\mathbf{\jmath}}. \tag{1}$$

Ex: If $f(x, y) = \sin x + e^{xy}$, then

$$\nabla f(x, y) = \langle f_x, f_y\rangle = \langle \cos x + ye^{x,y}, xe^{x,y}\rangle \Rightarrow \nabla f(0, 1) = \langle 2, 0\rangle.$$

Ex: Find the directional derivative of $f(x, y) = x^2y^3 - 4y$ at $(2, -1)$ in the direction of $\vec{v} = 2\hat{\mathbf{\imath}} + 5\hat{\mathbf{\jmath}}$.

Solution: Notice that v is not a unit vector, so lets take care of that first,

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{2}{\sqrt{29}}\hat{\mathbf{\imath}} + \frac{5}{\sqrt{29}}\hat{\mathbf{\jmath}}.$$

Next,

$$\nabla f(x, y) = 2xy^3\hat{\mathbf{\imath}} + (3x^2y^2 - 4)\hat{\mathbf{\jmath}} \Rightarrow \nabla f(2, -1) = -4\hat{\mathbf{\imath}} + 8\hat{\mathbf{\jmath}}.$$

Then $D_u f(2, -1) = \nabla f(2, -1) \cdot \vec{u} = 32/\sqrt{29}$.

It should be noted that these concepts are easily extended to multiple dimensions. So, in 3-D

$$\nabla f(x, y, z) = \langle f_x, f_y, f_z\rangle \tag{2}$$

and

$$D_u f(x, y, z) = \nabla f(x, y, z) \cdot \vec{u}. \tag{3}$$

Ex: If $f(x,y) = x\sin(yz)$,

(a) find the gradient of f

Solution:

$$\nabla f = \langle \sin(yz), xz\cos(yz), xy\cos(yz)\rangle.$$

(b) find the directional derivative of f at $(1,3,0)$ in the direction of $\vec{v} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$.

Solution: Again, $\vec{v}$ is not a unit vector.

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{6}}\hat{\mathbf{i}} + \frac{2}{\sqrt{6}}\hat{\mathbf{j}} - \frac{1}{\sqrt{6}}\hat{\mathbf{k}}.$$

and

$$\nabla f(1,3,0) = \langle 0,0,3\rangle \Rightarrow D_u f(1,3,0) = \nabla f(1,3,0)\cdot\vec{u} = -\frac{3}{2}.$$

Now suppose we want to find the fastest way to go up/down the hill, what direction do we choose? Another way to ask this is, what is the vector $\vec{u}$ that gives us the maximum dot product $\nabla f\cdot\vec{u}$.

Theorem 2. *Suppose f is a differentiable function, the maximum value of the directional derivative $D_u f$ is $\|\nabla f\|$ and it occurs when $\vec{u}$ is in the same direction as ∇f.*

Ex: (a) If $f(x,y) = xe^y$, find the rate of change of f at the point $P(2,0)$ in the direction from P to $Q(1/2,2)$.

Solution: $\vec{PQ} = \langle -3/2, 2\rangle$, then the unit direction vector is $\vec{u} = \langle -3/5, 4/5\rangle$ and

$$\nabla f = \langle e^y, xe^y\rangle \Rightarrow \nabla f(2,0) = \langle 1,2\rangle \Rightarrow D_u f(2,0) = \nabla f(2,0)\cdot\vec{u} = 1.$$

(b) In what direction does f have the maximum rate of change? What is the value of the maximum rate of change?

Solution: Direction of maximum rate: $\nabla f(2,0) = \langle 1,2\rangle$

Value of maximum rate: $\|\nabla f(2,0)\| = \|\langle 1,2\rangle\| = \sqrt{5}$.

Ex: Suppose the temperature in this room is given by $T(x,y,z) = 80/(1+x^2+2y^2+3z^2)$. In which direction does the temperature increase fastest at $(1,1,-2)$? What is this value?

Solution: Fastest:

$$\nabla T = \left.\frac{160x\hat{\mathbf{i}} - 320y\hat{\mathbf{j}} - 480z\hat{\mathbf{k}}}{(1+x^2+2y^2+3z^2)^2}\right|_{(1,1,-2)} = \frac{5}{8}(-\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}).$$

Value:

$$\|\nabla T\| = \frac{5}{8}\|\langle -1,-2,6\rangle = \frac{5\sqrt{41}}{8} \approx 4^\circ \quad \text{C/m}$$

If we go to 4-D just momentarily, we can find some really interesting properties of the gradient. Suppose a surface is defined by $F(x,y,z) = k$ (i.e., level surfaces), let $P(x_0,y_0,z_0)$ be a point on the surface, and consider a curve $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ through P; i.e., $\vec{r}(t_0) = \langle x_0,y_0,z_0\rangle$. Since $\vec{r}$ is on the surface, $F(x(t),y(t),z(t)) = k$, and if we take the derivative using chain rule we get

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0.$$

Notice that $\nabla F = \langle F_x, F_y, F_z\rangle$ and $\vec{r}\,' = \langle dx/dt, dy/dt, dz/dt\rangle$, then $\nabla F\cdot\vec{r}\,' = 0$. Specifically $\nabla F(x_0,y_0,z_0)\cdot\vec{r}\,' = 0$ for any curve on the surface; i.e., $\nabla F(x_0,y_0,z_0)$ is the normal vector for the tangent plane.

The equation of the tangent plane to a surface can then be written as

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0. \tag{4}$$

Ex: Find the equations of the tangent plane and normal line at point $(-2,1,-3)$ to the ellipsoid $x^2/4 + y^2 + z^2/9 = 3$.

Solution: Here $k = 3$, so $F(x,y,z) = x^2/4 + y^2 + z^2/9$, then $F_x = x/2$, $F_y = 2y$, $F_z = 2z/9$, so

$$\nabla F(-2,1,-3) = \langle -1, 2, -\frac{2}{3}\rangle \Rightarrow -(x+2) + 2(y-1) - \frac{2}{3}(z+3) = 0,$$

and the normal line in symmetric form is

$$\frac{x+2}{-1} = \frac{y-1}{2} = \frac{z+3}{-2/3}.$$

Notice that we can do this in 2-D too; i.e., the route of steepest ascent is always perpendicular to the contour lines.

11.7 Extrema

Definition 2. A function of two variables has a local maximum (respectively a local minimum) at (a,b) if $f(x,y) \leq f(a,b)$ (respectively $f(x,y) \geq f(a,b)$) when (x,y) is near (a,b). The number $f(a,b)$ is called a local maximum value (respectively local minimum value).

Theorem 3. *If f has a local maximum/minimum at (a,b) and f_x and f_y exist, then*

$$f_x(a,b) = f_y(a,b) = 0. \tag{5}$$

A way we can interpret this is to say the tangent plane is parallel to the xy-plane.

Ex: Let $f(x,y) = x^2 + y^2 - 2x - 6y + 14$. Notice that this is a paraboloid. Then $f_x(x,y) = 2x - 2$, $f_y(x,y) = 2y - 6$, and $f_x = f_y = 0$ when $x = 1$, $y = 3$. In order to use our definition lets put this into standard form by completing the square:

$$f(x,y) = 4 + (x-1)^2 + (y-3)^3$$

Notice that $f(1,3) = 4$ and since $(x-1)^2$ and $(y-3)^2$ are positive, $f(x,y) \geq 4$ for all x and y. Therefore, $(1,3)$ is a minimum.

Ex: Find the extreme values of $f(x,y) = y^2 - x^2$, or show it does not exist.

Solution: $f_x = -2x = 0$, $f_y = 2y = 0$, then $(0,0)$ is a critical point. However, on the x-axis for $x > 0$, $f(x,y) = -x^2 < 0$, and on the y-axis, if $y > 0$, $f(x,y) = y^2 > 0$. So, in any neighborhood around $(0,0)$ we have both $f < 0$ and $f > 0$. Therefore, no extrema exists. And this makes sense geometrically since the surface is a saddle.

But sometimes we don't want to do a geometric analysis. Is there an easier way?

Theorem 4 (Second derivative test)**.** *Suppose all second partial derivatives of f are continuous on a disk with center (a,b), and suppose that $f_x = f_y = 0$ (i.e., (a,b) is a critical point of f). Let the Hessian be*

$$H(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \tag{6}$$

then

(1) *If $H(a,b) > 0$ and $f_{xx}(a,b) > 0$, $f(a,b)$ is a local minimum value.*
(2) *If $H(a,b) > 0$ and $f_{xx}(a,b) < 0$, $f(a,b)$ is a local maximum value.*
(3) *If $H(a,b) < 0$, $f(a,b)$ is neither a maximum or a minimum value (e.g. saddle.)*