12.1 Double integration over rectangles

Since everyone knows how to integrate, lets jump right into it.

Ex: Estimate the volume of the solid that lies above the square $R = [0,2] \times [0,2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$ by dividing R into four equal squares and choose the sample points to be the upper right hand corners of the squares.

Solution: Let $z = f(x,y) = 16 - x^2 - 2y^2$, then

$$V \approx \sum_{i=1,\, j=1}^{2} f(x_i, y_j)\Delta A = f(1,1)\Delta A + f(1,2)\Delta A + f(2,1)\Delta A + f(2,2)\Delta A = 13\cdot 1 + 7\cdot 1 + 10\cdot 1 + 4\cdot 1 = \boxed{34}.$$

As we know we can make the approximation better by increasing the number of squares. However, it would be much easier if we could use the Fundamental Theorem of Calculus.

Ex: Evaluate

(a) $\int_0^3 \int_1^2 x^2 y\,dy\,dx$

Solution:

$$\int_1^2 x^2 y\,dy = x^2 \int_1^2 y\,dy = x^2 \left[\frac{1}{2}y^2\right]_1^2 = 2x^2 - \frac{1}{2}x^2 = \frac{3}{2}x^2.$$

then

$$\int_0^3 \frac{3}{2}x^2 dx = \frac{1}{2}x^3\Big|_0^3 = \boxed{\frac{27}{2}}.$$

(b) $\int_1^2 \int_0^3 x^2 y\,dx\,dy$

Solution: Similarly,

$$\int_1^2 \int_0^3 x^2 y\,dx\,dy = \int_1^2 y\left[\frac{1}{3}x^3\right]_0^3 dy = \int_1^2 9y\,dy = \frac{9}{2}y^2\Big|_1^2 = \boxed{\frac{27}{2}}.$$

Notice that both integrals are the same. While we can't always do this in real life, there is a theorem that will allow us to do this for the types of problems that we will see in this class.

Theorem 1 (Fubini). *If f is continuous on $R = \{(x,y) | a \le x \le b,\ c \le y \le d\}$, then*

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy. \tag{1}$$

It should be noted that this will also hold for removable discontinuities.

Ex: Evaluate $\iint_R (x - 3y^2)\,dA$ on $R = \{(a,b) | 0 \le x \le 2, 1 \le y \le 2\}$.

Solution:

$$\iint_R (x-3y^2)\,dA = \int_0^2 \int_1^2 (x - 3y^2)\,dy\,dx = \int_0^2 \left[xy - y^3\right]_1^2 dx = \int_0^2 (x-7)\,dx = \frac{x^2}{2} - 7x\Big|_0^2 = \boxed{-12}.$$

You should verify yourselves that it works the other way around.

Ex: Evaluate $\iint\limits_R y\sin(xy)dA$ on $R=[1,2]\times[0,\pi]$

Solution:

$$\iint\limits_R y\sin(xy)dA = \int_0^\pi \int_1^2 y\sin(xy)dxdy = \int_0^\pi \left[-\cos(xy)\right]_1^2 dy = \int_0^\pi (-\cos 2y + \cos y)dy = -\frac{1}{2}\sin 2y + \sin y\Big|_0^\pi = \boxed{0}.$$

If we reverse the order we get

$$\iint\limits_R y\sin(xy)dA = \int_1^2 \int_0^\pi y\sin(xy)dydx.$$

We have to do the inside integral "by parts". Use $u=y \Rightarrow du=dy$ and $dv=\sin(xy)dy \Rightarrow v=-\frac{1}{x}\cos(xy)$, which gives us

$$\int_0^\pi y\sin(xy)dy = -\frac{y}{x}\cos(xy)\Big|_0^\pi + \frac{1}{x}\int_0^\pi \cos(xy)dy = -\frac{\pi}{x}\cos(\pi x) + \frac{1}{x^2}\left[\sin(xy)\right]_0^\pi = -\frac{\pi}{x}\cos(\pi x) + \frac{1}{x^2}\sin(\pi x),$$

then

$$\iint\limits_R y\sin(xy)dA = \int_1^2 -\frac{\pi}{x}\cos(\pi x) + \frac{1}{x^2}\sin(\pi x)dx = \int_1^2 -\frac{\pi}{x}\cos(\pi x)dx + \int_1^2 \frac{1}{x^2}\sin(\pi x)dx.$$

Lets deal with the first integral first. We can do this one by parts if we have $u=-1/x \Rightarrow du=\frac{1}{x^2}dx$, and $dv=\pi\cos(\pi x)dx \Rightarrow v=\sin(\pi x)$, which gives us

$$\int -\frac{\pi}{x}\cos(\pi x)dx = -\frac{1}{x}\sin(\pi x) - \int \frac{1}{x^2}\sin(\pi x)dx \Rightarrow \int -\frac{\pi}{x}\cos(\pi x) + \frac{1}{x^2}\sin(\pi x)dx = -\frac{1}{x}\sin(\pi x),$$

so

$$\iint\limits_R y\sin(xy)dA = \int_1^2 \int_0^\pi y\sin(xy)dydx = -\frac{1}{x}\sin(\pi x)\Big|_1^2 = -\frac{1}{2}\sin(2\pi) + \sin(\pi) = \boxed{0}.$$

Which way was easier?

Ex: Find the volume of the solid bounded by $x^2+2y^2+z=16$ and planes $x=2$ and $y=2$ in the first octant (i.e., $z=y=x=0$).

Solution:

$$V = \int_0^2 \int_0^2 (16-x^2-2y^2)dxdy = \int_0^2 \left[16x - \frac{1}{3}x^3 - 2y^2x\right]_0^2 dy = \int_0^2 \left(\frac{88}{3} - 4y^2\right)dy = \left[\frac{88}{3}y - \frac{4}{3}y^3\right]_0^2 = \boxed{48}.$$

A bit of an aside, but there is a nice shortcut if you can factor out equations of only x and only y: if $f(x,y)=g(x)h(y)$,

$$\int_a^b \int_c^d g(x)h(y)dydx = \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right). \tag{2}$$

Ex:

$$\int_0^{\pi/2}\int_0^{\pi/2} \sin x\cos y dxdy = \left(\int_0^{\pi/2}\sin x dx\right)\left(\int_0^{\pi/2}\cos y dy\right) = \left(-\cos x\Big|_0^{\pi/2}\right)\left(\sin y\Big|_0^{\pi/2}\right) = \boxed{1}.$$

12.2 Double integration over general regions

What if two of the sides of our rectangle wasn't straight, but rather functions of x or y. Lets look at the two types of regions we could get.

Type 1: Boundary of region D are functions of only x;

$$D = \{(x,y) \quad | \quad a \le x \le b,\ g_1(x) \le y \le g_2(x)\} \Rightarrow \iint_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx. \tag{3}$$

Type 2: Boundary of region D are functions of only y;

$$D = \{(x,y) \quad | \quad c \le y \le d,\ h_1(x) \le x \le h_2(x)\} \Rightarrow \iint_D f(x,y)dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)dydx. \tag{4}$$

Ex: Evaluate $\iint_D (x+2y)dA$, where D is bounded by $y = 2x^2$ and $y = 1 + x^2$.

Solution: We first look for the intersection points: $2x^2 = 1 + x^2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$, so the points of intersection are $(-1, 2)$ and $(1, 2)$. Then our integral is

$$I = \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)dydx = \int_{-1}^{1} \left[xy + y^2\right]_{2x^2}^{1+x^2} = \int_{-1}^{1} \left[x(1+x^2) + (1+x^2)^2 - x(2x^2) - (2x^2)^2\right] dx$$

$$= \int_{-1}^{1} \left(-3x^4 - x^3 + 2x^2 + x + 1\right) dx = -3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x\Big|_{-1}^{1} = \boxed{\frac{32}{15}}.$$

Ex: Find the volume under $z = x^2 + y^2$ and above the region in the xy-plane bounded by $y = 2x$ and $y = x^2$.

Solution: We again look for the points of intersection, which we can see is $(0, 0)$ and $(2, 4)$.

$$V = \int_0^2 \int_{x^2}^{2} x(x^2+y^2)dydx = \int_0^2 \left[x^2 y + \frac{y^3}{3}\right]_{x^2}^{2x} dx = \int_0^2 \left[x^2(2x) + \frac{(2x)^3}{3} - x^2(x^2) - \frac{(-x^2)^3}{3}\right] dx$$

$$= \int_0^2 \left(-\frac{x^6}{3} - x^4 + \frac{14x^3}{3}\right) dx = -\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6}\Big|_0^2 = \boxed{\frac{216}{35}}.$$

Notice that the boundary can also be functions in the x-direction.

Often when we have square roots we may need to split the equation between the positive and negative parts, so we avoid it if we can, but here our intersection points are only in the first quadrant, so we are safe.

$$V = \int_0^4 \int_{y/2}^{\sqrt{y}} (x^2+y^2)dxdy = \int_0^4 \left[\frac{x^3}{3} + y^2 x\right]_{y/2}^{\sqrt{y}} dy = \int_0^4 \left(\frac{y^{3/2}}{3} + y^{5/2} - \frac{y^3}{24} - \frac{y^3}{2}\right) dy$$

$$= \frac{2}{15}y^{5/2} + \frac{2}{7}y^{7/2} - \frac{13}{96}y^4\Big|_0^4 = \boxed{\frac{216}{35}}.$$