LECTURE ONE: REVIEW OF LINEAR ALGEBRA AND ORDINARY
DIFFERENTIAL EQUATIONS

- Differential equations review topics:
  - Direction fields
  - Solving first order equations
  - Modeling (first and second order equations)
  - Systems of first order equations
  - Nonautonomous ODEs
- Linear algebra review topics:
  - Multiplying matrices
  - Taking determinants
  - Linear independence
  - Eigenvalues
  - Eigenvectors

We will speak in terms of first order autonomous ODEs because second order ODEs can be reduced into a system of first order ODEs and systems of nonautonomous ODEs can be reduced to systems of autonomous ODEs. For some simple examples please refer to the youtube video. I have also provided some simple examples of linearly independent/dependent vectors. Bellow we do some more interesting modeling examples employing most of the topics.

Lets now do a couple of modeling examples,

Ex: Consider a tank containing 100 Liters of water. There is a salt mixture entering at a rate of 300 Liters per hour with a concentration of one gram of salt per 100 Liters mixture. The mixture in the tank is leaving at the same rate as the incoming mixture is entering. Lets model, solve, and analyze (via a direction field) the amount of salt in the mixture.
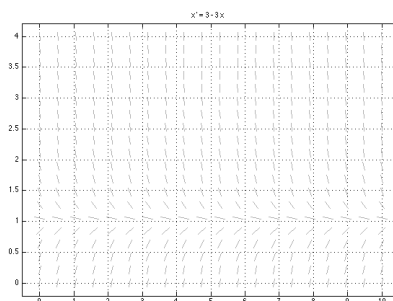
First we model it. Let the amount of salt inside the tank at any given time t be denoted by x. Notice that the salt is entering at a rate of 3 grams per hour, and the salt is leaving at a rate of $3x$ grams per hour. Then our model is,

$$\dot{x} = 3 - 3x. \tag{1}$$

We solve this via seperation to get,

$$x = 1 - e^{-3t}. \tag{2}$$

And we analyze the direction field,

Ex: Consider a mass on a spring system where the spring is attached to a horizontal platform and the mass hangs down from it. The mass weighs 32 lb and stretches the spring (under gravity) to 4 ft from its natural length. Suppose the mass is under the effect of a viscous damper such that it's terminal velocity is 8 ft/s. Lets model this, then convert the second order equation into a system of two first order equations. Next, lets find the eigenvalues and eigenvectors for this second order system.

First we model the system using Newton's second law,

$$m\ddot{x} + \gamma\dot{x} + kx = \ddot{x} + 4\dot{x} + 8x = 0. \tag{3}$$

We can solve this quite easily, however in order to use some of the techniques involved in analyzing more interesting problems we forgo solving it. Instead we reduce this second order equation into a system of first order equations by letting $v = \dot{x}$,

$$\dot{x} = v, \tag{4}$$

$$\dot{v} = -4v - 8x; \tag{5}$$

Now we can find the eigenvalues and eigenvectors,

$$\begin{vmatrix} -\lambda & 1 \\ -8 & -4-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 4\lambda + 8 = 0 \Rightarrow \lambda = -2 \pm 2i.$$

To find the eigenvectors recall we plug the eigenvalue back into the eigenvalue problem and solve for x,

$$\begin{pmatrix} 2+2i & 1 \\ -8 & -2+2i \end{pmatrix} x^{(-)} = 0 \Rightarrow x^{(-)} = \begin{pmatrix} -1 \\ 2+2i \end{pmatrix}$$

and

$$\begin{pmatrix} 2-2i & 1 \\ -8 & -2-2i \end{pmatrix} x^{(+)} = 0 \Rightarrow x^{(+)} = \begin{pmatrix} -1 \\ 2-2i \end{pmatrix}$$