

LECTURE TEN: FRACTALS

Cardinality of sets. We looked at some examples of finite sets in class, but what happens when two sets are infinite. Is one infinity bigger than the other? Notice that we can count the natural numbers, but we can't count say the real numbers, so there must be a difference between these two sets.

Definition 1. A set \mathbb{S} is said to be countably infinite if there is a one-to-one function from \mathbb{N} to \mathbb{S} .

So the question we must ask when we are thinking about infinities is, can we write the elements of \mathbb{S} as a sequence? Notice for $0, 1, 2, 3, \dots$ our function is, $f(n) = n - 1$ for $n \in \mathbb{N}$.

Definition 2. A set \mathbb{S} is said to be countable if it is either finite or countably infinite, otherwise it is said to be uncountable.

Notice that the reals are uncountable. Now to prove this takes a bit of effort, but we can intuitively see it when we try to "count" the reals.

Cantor set. A Cantor set is any set homeomorphic to $C = \prod_{n=1}^{\infty} F_n$, where each F_n is the two-point space $\{0,1\}$. This is however a complicated definition, and requires a knowledge of topology. Instead of defining it topologically it is useful to construct the set and present pictorial examples.

The Cantor set is constructed from the set of Real numbers in the unit interval $[0, 1]$. Lets call the initial set S_0 . For the first iteration a fraction α is taken away from S_0 such that S_1 contains two disconnected sets of Real numbers $[0, \frac{1}{2}(1 - \alpha)]$ and $[1 - \frac{1}{2}(1 - \alpha), 1]$. Lets $a := \frac{1}{2}(1 - \alpha)$ and $b := 1 - \frac{1}{2}(1 - \alpha)$, and call $[0, a]$ S_{1a} and $[b, 1]$ S_{1b} . For the second iteration take away the fraction α from S_{1a} and S_{1b} such that S_2 contains four disconnected sets of Real numbers $[0, \frac{1}{2}(a - \alpha a)]$, $[a - \frac{1}{2}(a - \alpha a), a]$, $[b, b + \frac{1}{2}(a - \alpha a)]$, and $[1 - \frac{1}{2}(a - \alpha a), 1]$. This is continued until S_{∞} is reached, and S_{∞} is called the Cantor set; more specifically the middle - α Cantor set. Two illustrations are shown in Figure .



FIGURE 1. Example of set S_7 . Wikipedia.

For the sake of rigor a closed form formula for each iteration is needed. It is easy to see $S_1 = \frac{1}{2}(1 - \alpha)S_0 \cup [\frac{1}{2}(1 + \alpha) + \frac{1}{2}(1 - \alpha)S_0]$. The formula for S_2, S_3, etc may seem somewhat difficult to derive, but with a little inspection and some computations a repeating pattern is seen. Notice $S_2 = \frac{1}{2}(1 - \alpha)S_1 \cup [\frac{1}{2}(1 + \alpha) + \frac{1}{2}(1 - \alpha)S_1]$ and $S_3 = \frac{1}{2}(1 - \alpha)S_2 \cup [\frac{1}{2}(1 + \alpha) + \frac{1}{2}(1 - \alpha)S_2]$. We may assume the formula for the n^{th} case follows this pattern.

$$S_n = \frac{1}{2}(1 - \alpha)S_{n-1} \cup \left[\frac{1}{2}(1 + \alpha) + \frac{1}{2}(1 - \alpha)S_{n-1}\right] \quad (1)$$

$$0 < \alpha < 1, S_0 = [0, 1], \quad n \neq \infty$$

The proof of n arbitrarily large is shown, however for $n = \infty$ a more rigorous proof is required.

Proof. By definition

$$S_0 = [0, 1].$$

It is easy to see

$$S_1 = \frac{1}{2}(1 - \alpha)S_0 \cup \left[\frac{1}{2}(1 + \alpha) + \frac{1}{2}(1 - \alpha)S_0\right] = \left[0, \frac{1}{2}(1 - \alpha)\right] \cup \left[1 - \frac{1}{2}(1 - \alpha), 1\right],$$

is true.

Assume

$$S_n = \frac{1}{2}(1 - \alpha)S_{n-1} \cup \left[\frac{1}{2}(1 + \alpha) + \frac{1}{2}(1 - \alpha)S_{n-1}\right],$$

is true.

It can be shown

$$S_{n+1} = \frac{1}{2}(1 - \alpha)S_n \cup \left[\frac{1}{2}(1 + \alpha) + \frac{1}{2}(1 - \alpha)S_n\right].$$

Let

$$\beta_n = \frac{1}{2}(1 - \alpha)\beta_{n-1} \cup \left[\frac{1}{2}(1 + \alpha) + \frac{1}{2}(1 - \alpha)\beta_{n-1}\right], \quad (2)$$

and

$$\beta_{n-1} = S_n = \frac{1}{2}(1 - \alpha)S_{n-1} \cup \left[\frac{1}{2}(1 + \alpha) + \frac{1}{2}(1 - \alpha)S_{n-1}\right] \quad (3)$$

Plugging Equation 3 into Equation 2 gives,

$$\beta_n = \frac{1}{2}(1 - \alpha)S_n \cup \left[\frac{1}{2}(1 + \alpha) + \frac{1}{2}(1 - \alpha)S_n\right] = S_{n+1}$$

By induction

$$S_n = \frac{1}{2}(1 - \alpha)S_{n-1} \cup \left[\frac{1}{2}(1 + \alpha) + \frac{1}{2}(1 - \alpha)S_{n-1}\right]$$

$$S_0 = [0, 1], \quad n \neq \infty$$

thereby completing the proof. \square

Some Important Properties of Cantor Sets.

- (1) Cantor sets are self-similar fractals.
 - Cantor sets look the same no matter the level at which they are seen. All $n + 1$ sections of S_n looks the same as S_0 when magnified.
- (2) Cantor sets are completely disconnected.
 - There are no intervals within a Cantor set (i.e. it has a topological dimension of zero, however it has a nonzero fractal dimension).
- (3) Cantor sets have a measure of zero.
 - The length of S_n is α^n . When $n = \infty$ the length of S_n is zero, because $0 < \alpha < 1$.

- Another way to find the measure is to subtract the measure of the complement of the Cantor set from the total length of $[0,1]$. The length of the complement of the Cantor set is

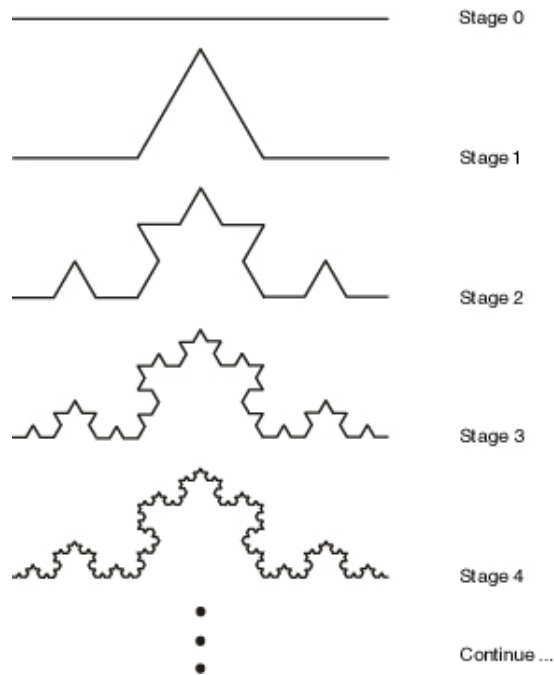
$$\sum_{n=0}^{\infty} \left(\frac{1}{\alpha} - 1\right)^n \alpha^{n+1} = \alpha \sum_{n=0}^{\infty} \left(\frac{1}{\alpha} - 1\right)^n \alpha^n = \alpha \sum_{n=0}^{\infty} (1 - \alpha)^n = \frac{\alpha}{\alpha} = 1 \quad (4)$$

(4) Cantor sets are uncountable.

- This can be shown by using Cantor's diagonal argument.

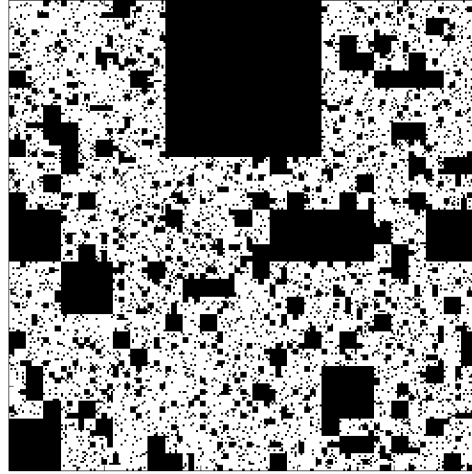
Dimension of self-similar fractals. Notice that the Cantor set seems one dimensional, but has more structure than a one dimensional object. But of course, it's not two dimensional. How do we rectify this? For a one dimensional object for example, we only have one copy and it's never scaled down. For a self similar fractal at each iteration we make more copies and they are all scaled down. Take the Cantor set for example, we make $n = 2$ copies and scale them down by $r = 3$, so we can relate how the copies scale down as $n = r^d$, then $d = \ln(n)/\ln(r)$, then the dimension of the Cantor set is $d = \ln(2)/\ln(3)$, so it's between one and two dimensions.

In class we also discussed the von Koch curve shown bellow,



Notice that this makes $n = 4$ copies and scales by a factor of $r = 3$, so it will be between one and two dimensions: $d = \ln(4)/\ln(3)$.

Box dimension. The idea of the box dimension is to cover the entire set with the minimum number of boxes of size ε . Let N be the number of boxes, then $d = \lim_{\varepsilon \rightarrow 0} \ln N / \ln(1/\varepsilon)$. This works well for, say the non self similar version of the Sierpinski carpet, shown bellow, but rarely works in practice for general fractals.



Fractals in chaos. In phase space chaotic systems very typically have fractal structure. For example, strange attractors, horseshoes, period doubling, and others all have fractal structure. One way we can think of the dimension of an object is the number of times a trajectory intersects balls around various points. This is called the correlation dimension, which we discussed in class. However, as we discussed this can only give us an approximation. There are also sets where the dimension varies from one region to another, this is called multifractals, but we won't get into it in this class.